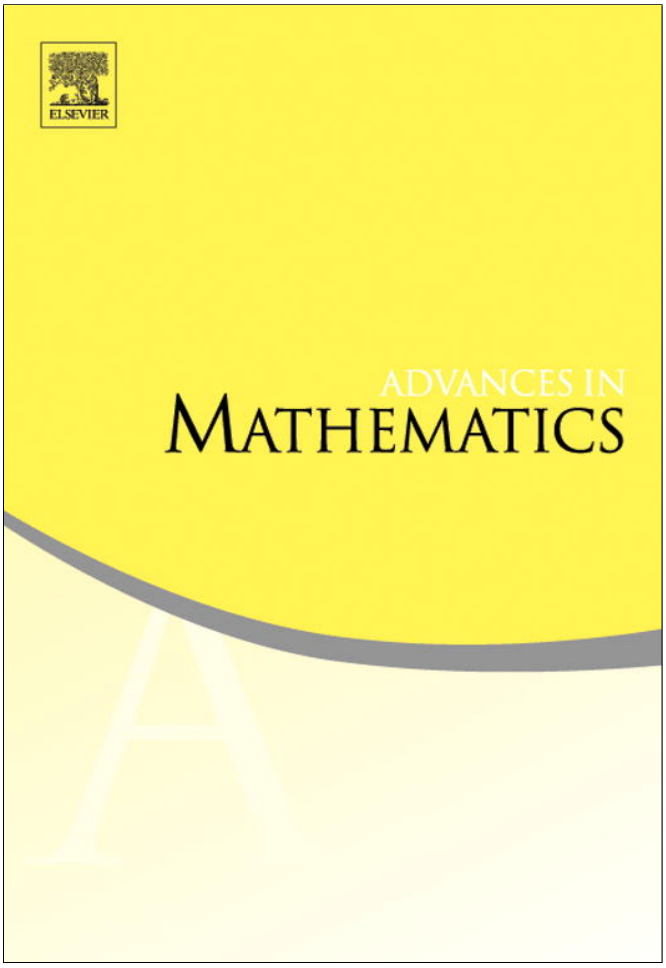

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Distributionally concave symmetric spaces and uniqueness of symmetric structure

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Abstract

We present new results concerning the uniqueness of symmetric structure of symmetric function spaces. Our methods are partly based on a detailed study of distributionally concave spaces and the tensor product operator.

Keywords: Symmetric space; Orlicz space; Orlicz–Lorentz space; Distributionally concave space; Uniqueness of symmetric structure; Tensor product operator

1. Introduction

To better explain the motivation and main results of this paper, we first cite the following important result from the Memoir of Johnson et al. [15], which extends a well-known result that every $L_p[0, 1]$-space, $1 \le p \le \infty$ has a unique representation as a symmetric space on $[0, 1]$ [24, 2.e.8].

Theorem 1.1 *([15, Corollary 7.9]). Let a separable Orlicz space $L_M = L_M[0, 1]$ be p-convex for some $p > 2$. Then the following conditions are equivalent.*

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(1) *If a symmetric function space X is isomorphic to a subspace of the space L_M, then either $X = L_2[0,1]$ or $X = L_M[0,1]$ (up to norm equivalence).*
(2) *For some $C > 0$ and any $u, v \geq 1$ the following holds*

$$M(uv) \leq CM(u)M(v). \tag{1.1}$$

In particular, the condition (1.1) provides the uniqueness of symmetric structure of a p-convex ($p > 2$) separable Orlicz space L_M (that is, an arbitrary symmetric space X on $[0,1]$ which is isomorphic to L_M can be equivalently renormed so that $X = L_M$). Comparable results were also achieved for Lorentz spaces $\Lambda(p, \psi)$, $p > 2$ [13,14].

Our analysis of the proof of Theorem 1.1 in [15] and Carothers' results in [13,14] have led us to believe that they are all underpinned by a certain geometric property of symmetric function spaces, which we have termed "the distributional concavity" in this paper. In the present paper the result of Theorem 1.1 is extended to a much wider class of distributionally concave symmetric function spaces, which contains both Orlicz and Lorentz spaces (moreover, it contains the class of so-called Orlicz–Lorentz spaces). The distributional concavity was introduced by Montgomery-Smith and Semenov in [32][1] (see also an alternative approach to this property in [38] and discussion in the appendix to [28]). This property is a natural counterpart of the notion of distributional convexity introduced originally by Kalton in [16]. The class of distributionally concave symmetric function spaces appears to be of interest in its own right; however, the treatment of this class in [32] is rather patchy. In particular, the end of the proof of Theorem 22 in [32] (which provides an important characterization of this class) contains a mistake. In this paper, we rectify this mistake and provide the complete proof of that theorem together with a number of corollaries. Using this opportunity, we present in the last section a detailed exposition of distributionally concave symmetric function spaces and its properties. We thank Professors Montgomery-Smith and Semenov for numerous discussions concerning [32] and their encouragement to publish our account of their results.

The main results are Theorems 1.2 and 1.3 and Corollaries 1.4 and 1.5 below. Another important technical advantage of our approach to the isomorphic classification of symmetric spaces is the consistent usage of the tensor product operator (which is a well-known tool in interpolation theory). The latter operator lurks in the background of a number of proofs in [24,15,13,14]; however, its importance in the isomorphic theory of symmetric function spaces appears to have been overlooked. Indeed, in the class of symmetric spaces which is studied in this paper, the boundedness of this operator is, in fact, equivalent to the uniqueness of symmetric structure as evidenced from Corollary 1.4 below.

Theorem 1.2. *Let E be a distributionally concave separable symmetric function space on* $[0,1]$*, such that*

(a) *E is an interpolation space for the couple (L_2, L_∞)*
and
(b) *the tensor product operator $(x, y) \to x \otimes y$ is bounded from the Cartesian square $E \times E$ into E.*

For every symmetric space F isomorphic to a subspace of E, one of the following options holds.

(1) *Either $F = L_2[0,1]$ or $F = E$.*
(2) *The Haar system in F is equivalent to a sequence of disjointly supported functions in E.*

[1] termed D^*-convexity there

Theorem 1.3. *If E is a distributionally concave separable symmetric function space on* $[0, 1]$*, such that E satisfies an upper p-estimate for some $p > 2$, then the following conditions are equivalent.*

(1) *Spaces* $L_2[0, 1]$ *and E are the only symmetric spaces on* $[0, 1]$ *isomorphic to a subspace of E.*
(2) *The tensor product operator $x \otimes y$ is bounded from the Cartesian square $E \times E$ into E.*

Corollary 1.4. *If a symmetric function space E satisfies the assumptions of Theorem 1.3, then its symmetric structure is unique if and only if the tensor product operator is bounded from the Cartesian square $E \times E$ into E.*

Corollary 1.5. *Let M be a p-convex (for some $p > 2$) Orlicz function satisfying condition (1.1) and let ψ be a concave increasing on* $[0, 1]$ *function, such that $\psi(0) = 0$ and for some $C' > 0$*

$$\psi(st) \le C'\psi(s)\psi(t), \quad 0 < s, t \le 1. \tag{1.2}$$

Then $L_2[0, 1]$ *and the Orlicz–Lorentz space $\Lambda(M, \psi)$ are the only symmetric spaces on* $[0, 1]$ *isomorphic to a subspace of $\Lambda(M, \psi)$. In particular, $\Lambda(M, \psi)$ has a unique symmetric structure.*

It is well known that any Orlicz space L_M is distributionally concave [16,32,38] and that the tensor product operator on $L_M \times L_M$ is bounded into L_M if and only if the function M satisfies condition (1.1) [1, Theorem 6]. Thus, Theorem 1.1 is an immediate corollary of Theorem 1.3. It should be also pointed out that Theorem 1.2 immediately implies earlier results from [13,14] concerning the uniqueness of symmetric structure of Lorentz spaces $\Lambda(p, \psi)$ (indeed, the Haar system in a symmetric space F cannot be equivalent to a sequence of pairwise disjoint functions in $\Lambda(p, \psi)$; see [13, Lemma 1 and arguments preceding it]).

To make our exposition straightforward, the proof of [32, Theorem 22] (see Theorem 4.4 below) and its corollaries is postponed until the last section. However, the results from that section (in particular, Theorem 4.6) are used in the proof of Theorem 1.2 in a crucial way.

We will prove all our main results in Section 3. Section 2 is devoted to the preliminary information required for the proofs in Section 3.

2. Preliminaries

2.1. Symmetric spaces

Let $L_0 = L_0(I)$ be the space of finite almost everywhere Lebesgue measurable functions either on $I = [0, 1]$ or on $I = [0, \infty)$ (with identification $m-$a.e.). Here m is the Lebesgue measure. Denote by $S_0 = S_0(I)$ the subset of L_0 which consists of all functions x such that *the distribution function*

$$d_x(s) := m(\{t \in I : |x(t)| \ge s\})$$

is finite for some $s > 0$. Any two functions x and y from S_0 are said to be *equimeasurable* if $d_x(s) = d_y(s)$ for every $s > 0$.

Let E be a Banach space of real-valued Lebesgue measurable functions either on $[0, 1]$ or $[0, \infty)$. The space E is said to be an *ideal lattice* if the conditions $x \in E$ and $|y| \le |x|$, $y \in L_0$, imply that $y \in E$ and $\|y\|_E \le \|x\|_E$.

The ideal lattice $E \subseteq S_0(I)$ is said to be a *symmetric space* on I if the norm $\|\cdot\|_E$ is symmetric, that is, for every $x \in E$ and every equimeasurable with x function $y \in L_0$ (equivalently, $y^* = x^*$), we have $y \in E$ and $\|y\|_E = \|x\|_E$ (see [22,24]).

Here, x^* denotes the non-increasing right-continuous rearrangement of x given by

$$x^*(t) := \inf\{s \geq 0 : \ d_x(s) \leq t\}.$$

Following [15] (see also [24, Section 2.f]), for each symmetric space E on [0, 1] we define Z_E^2 as the set of all measurable on $(0, \infty)$ functions f such that

$$\|f\|_{Z_E^2} := \|f^* \chi_{[0,1]}\|_E + \|f^* \chi_{[1,\infty)}\|_{L_2[1,\infty)} < \infty.$$

It can easily be shown that the quasinorm $\|\cdot\|_{Z_E^2}$ is equivalent to a symmetric norm, so that Z_E^2 is a symmetric space on $[0, \infty)$.

Let E be a symmetric space on [0, 1]. Denote by $E(l_2)$ the completion of the set of all eventually vanishing sequences $\{x_k\}_{k=1}^\infty$ of functions from E with respect to the norm

$$\|\{x_k\}\|_{\widetilde{E(l_2)}} := \sup_{n=1,2,\ldots} \left\| \left(\sum_{k=1}^n x_k^2 \right)^{1/2} \right\|_E .$$

Define the tensor product operator from the Cartesian product $L_0[0, 1] \times L_0[0, 1]$ to $L_0([0, 1]^2)$ by setting

$$(x, y) \to x \otimes y : (x \otimes y)(s, t) := x(s)y(t), \quad s, t \in [0, 1].$$

Similarly, the tensor product operator $(x, y) \to x \otimes y$ is defined in the case when $x \in L_0[0, 1]$ and $y \in L_0[0, \infty)$. Observe that it follows immediately from the Closed Graph Theorem that the tensor product operator sends the Cartesian product $E \times E$ into E if and only if it is a bounded mapping from the Banach space $E \times E$ (equipped with the product topology) into the Banach space E.

For a symmetric space E on I the *Köthe dual* E' is the space of all $f \in S_0(I)$ such that the *associate norm*

$$\|f\|_{E'} := \sup_{g \in E, \ \|g\|_E \leq 1} \int_I |f(x)g(x)| \, dx$$

is finite. The Köthe dual $E' = (E', \|\cdot\|_{E'})$ is a symmetric space. We set $E'' := (E')'$. Obviously, $E \subset E''$ with $\|f\|_E \geq \|f\|_{E''}$ for all $f \in E$. If $\|f\|_E = \|f\|_{E''}$ for all $f \in E$, then the norm of E is called *order semicontinuous.* A symmetric space E on I has an order semicontinuous norm if and only if E has the following property: if $f_n, f \in X$ and $0 \leq f_n \nearrow f$ a.e. on I, then $\|f_n\| \nearrow \|f\|$. When $E'' = E$ we say that a symmetric space E *has the Fatou property*. This is equivalent to the following: if $f_n \in E$, $f \in L_0$, $0 \leq f_n \nearrow f$ a.e. on I and $\sup_{n \in \mathbb{N}} \|f_n\|_E < \infty$, then $f \in E$ and $\|f_n\| \nearrow \|f\|$.

For more details on the properties of symmetric spaces see, for example, [22,24].

2.2. *Examples of symmetric spaces*

Important examples of symmetric spaces are given by Lorentz and Orlicz spaces. Let ψ be an increasing concave function on I with $\psi(0) = \psi(+0) = 0$, $1 \leq p < \infty$. The Lorentz space

$\Lambda(p,\psi)$ on I consists of all measurable functions f on I for which

$$\|f\|_{\Lambda(p,\psi)} := \left(\int_I f^*(t)^p\, d\psi(t)\right)^{1/p} < \infty.$$

In particular, $\Lambda(\psi) := \Lambda(1,\psi)$.

Let M be an Orlicz function on $[0,\infty)$, that is, M is a continuous convex increasing function on $[0,\infty)$ satisfying $M(0) = 0$ and $M(\infty) = \infty$. Then the space $L_M(I)$ is the set of all Lebesgue measurable functions f on I for which

$$\int_I M\left(\frac{|f(t)|}{\rho}\right) dt < \infty$$

for some $\rho > 0$. The (Luxemburg) norm on $L_M(I)$ is defined by

$$\|f\|_{L_M} := \inf\left\{\rho > 0 : \int_I M\left(\frac{|f(t)|}{\rho}\right) dt \le 1\right\}.$$

A natural generalization of Orlicz and Lorentz spaces is the Orlicz–Lorentz spaces, $\Lambda(M,\psi)$ (see e.g. [18,19]). Let ψ be a positive concave function on I with $\psi(0+) = 0$ and M be an Orlicz function on $[0,\infty)$. Define the functional ρ on $S_0(I)$ by the formula: $\rho(f) := \int_I M(f^*(s))\, d\psi(s)$. Then

$$\Lambda(M,\psi) := \left\{f \in S_0 : \rho\left(\frac{f}{\lambda}\right) < \infty \text{ for some } \lambda > 0\right\}.$$

The space $\Lambda(M,\psi)$ is a symmetric space on I with respect to the norm

$$\|f\|_{\Lambda(M,\psi)} := \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Note that if we take $\psi(t) = t$ (respectively, $M(t) = t$), then $\Lambda(M,\psi)$ is the Orlicz space L_M (respectively, Lorentz space $\Lambda(\psi)$).

2.3. *Boyd indices*

Let E be a symmetric space on I. We define the dilation operator $\sigma_a : E \to E$ by

$$\sigma_a x(t) = x\left(\frac{t}{a}\right), \quad a > 0 \text{ on } [0,\infty)$$

while for $[0,1]$, we have

$$\sigma_a x(t) = \begin{cases} x\left(\frac{t}{a}\right) & \text{for } 0 \le t \le \min\{1,a\} \\ 0 & \text{for } \min\{1,a\} < t \le 1. \end{cases}$$

We use the definition of Boyd indices as in [24, Definition 2.b.1, p. 130].

Definition 2.1. Let E be a symmetric function space on $[0,1]$ or $[0,\infty)$. The Boyd indices p_E and q_E are defined by

$$p_E = \lim_{s\to\infty} \frac{\log s}{\log \|\sigma_s\|_{E\to E}} = \sup_{s>1} \frac{\log s}{\log \|\sigma_s\|_{E\to E}}$$

$$q_E = \lim_{s\to 0^+} \frac{\log s}{\log \|\sigma_s\|_{E\to E}} = \inf_{0<s<1} \frac{\log s}{\log \|\sigma_s\|_{E\to E}}.$$

Note that if $E = L_p(I)$, $1 \le p \le \infty$, then $p_E = q_E = p$.

2.4. Distributional convexity and concavity

Given functions $x_1, x_2, \ldots, x_n$ on [0, 1], we define their dilated disjoint sum to be the function on [0, 1] given by

$$C(x_1, \ldots, x_n)(t) = x_k(nt - k + 1) \quad \text{for } \frac{k-1}{n} \le t < \frac{k}{n},\ k = 1, \ldots, n.$$

Definition 2.2. A symmetric space E on the segment [0, 1] is called distributionally convex if there is a constant $c_E > 0$ such that

$$\|C(x_1, \ldots, x_n)\|_E \le c_E \max_{1\le k\le n} \|x_k\|_E$$

and distributionally concave if there is a constant $c'_E > 0$ such that

$$\|C(x_1, \ldots, x_n)\|_E \ge (c'_E)^{-1} \min_{1\le k\le n} \|x_k\|_E\ .$$

We may interpret these notions geometrically. Let V be the vector space of right-continuous functions $f : [0, \infty) \to \mathbb{R}$ of bounded variation. Define the following subsets of V:

$$B_c^{\le} = \{d_x : \|x\|_E \le c\} \quad \text{and} \quad B_c^{\ge} = \{d_x : \|x\|_E \ge c\}.$$

The following proposition (stated without proof in [32]) gives a geometric interpretation of distributional convexity (concavity) properties. By $\mathrm{conv}(A)$ we denote the convex hull of the set A.

Proposition 2.3. *A symmetric space E is distributionally concave (respectively, distributionally convex) if and only if there is a constant $c > 0$ such that* $\mathrm{conv}(B_1^{\ge}) \subseteq B_c^{\ge}$ *(respectively,* $\mathrm{conv}(B_1^{\le}) \subseteq B_c^{\le}$*).*

Proof. We prove the statement for the case of distributionally concave spaces. The proof for the distributionally convex case is quite similar.

First suppose that there is a constant $c > 0$ such that $\mathrm{conv}(B_1^{\ge}) \subseteq B_c^{\ge}$. We shall prove that

$$\|C(x_1, \ldots, x_n)\|_E \ge c \min_{1\le k\le n} \|x_k\|_E\ .$$

Clearly, we may assume that $\min_{1\le k\le n} \|x_k\|_E > 0$ (otherwise, there is nothing to prove). In this case, for any non-zero elements $x_1, \ldots, x_n \in E$, we set

$$y_k := \frac{x_k}{\min\limits_{1\le i\le n} \|x_i\|_E}, \quad k = 1, \ldots, n.$$

Since $\|y_k\|_E \ge 1$, we have $d_{y_k} \in B_1^{\ge}$. Hence,

$$d_{C(y_1,\ldots,y_n)} = \frac{1}{n} \sum_{k=1}^{n} d_{y_k} \in \mathrm{conv}(B_1^{\ge}) \subseteq B_c^{\ge}.$$

Therefore,

$$\|C(y_1,\ldots,y_n)\|_E \geq c, \text{ that is } \|C(x_1,\ldots,x_n)\|_E \geq c \min_{1\leq k\leq n} \|x_k\|_E .$$

Now suppose that E is distributionally concave with a constant c'_E. Fix $n \geq 1$ and let $x_1,\ldots,x_n \in E$ be such that $\|x_k\|_E \geq 1$ for all $k = 1,\ldots,n$. Consider a function

$$f := \sum_{k=1}^{n} \lambda_k d_{x_k} \in \operatorname{conv}(B_1^{\geq}),$$

where $\lambda_k \geq 0$ and $\sum_{k=1}^n \lambda_k = 1$. A convex combination of distribution functions is also a distribution function; so $f = d_y$ for some $y \in S_0[0,1]$. Let us show that $\|y\|_E \geq (c'_E)^{-1}$.

Without loss of generality, we may suppose $\lambda_k = a_k/m$, where $a_k \in \mathbb{N}$, $\sum_{k=1}^n a_k = m$ for some $m \in \mathbb{N}$. Then

$$d_y = \sum_{k=1}^{n} \lambda_k d_{x_k} = \frac{1}{m}\left(\overbrace{d_{x_1}+\cdots+d_{x_1}}^{a_1 \text{ of them}} + \cdots + \overbrace{d_{x_n}+\cdots+d_{x_n}}^{a_n \text{ of them}}\right),$$

which implies that

$$y = C(\overbrace{x_1,\ldots,x_1}^{a_1 \text{ of them}},\ldots,\overbrace{x_n,\ldots,x_n}^{a_n \text{ of them}})$$

Therefore,

$$\begin{aligned}\|y\|_E &= \|C(x_1,\ldots,x_1,\ldots,x_n,\ldots,x_n)\|_E \\ &\geq (c'_E)^{-1} \min_{1\leq k\leq n} \|x_k\|_E \quad \text{(since } E \text{ is distributionally concave)} \\ &\geq (c'_E)^{-1} \quad \text{(since } \|x_k\|_E \geq 1 \text{ for all } k = 1,2,\ldots,n). \quad \square\end{aligned}$$

Orlicz spaces L_M are both distributionally convex and concave with $c_{L_M} = c'_{L_M} = 1$ [38, Proposition 2.4]. In fact, a direct argument easily shows that

$$\min_{1\leq k\leq n} \|x_k\|_{L_M} \leq \|C(x_1,\ldots,x_n)\|_{L_M} \leq \max_{1\leq k\leq n} \|x_k\|_{L_M} .$$

An immediate corollary of the following assertion (see [38, Proposition 2.5]) is that every Lorentz space $\Lambda(\psi)$ is distributionally concave with $c'_{\Lambda(\psi)} = 1$.

Lemma 2.4. *We have*

$$\|C(x_1,\ldots,x_n)\|_{\Lambda(\psi)} \geq \frac{1}{n}\sum_{k=1}^{n} \|x_k\|_{\Lambda(\psi)}$$

for arbitrary sequences $\{x_k\}_{1\leq k\leq n} \subset \Lambda(\psi)$.

We shall now show that every Orlicz–Lorentz space is also distributionally concave.

Proposition 2.5. *Every Orlicz–Lorentz space* $\Lambda(M,\psi)$ *on* $[0,1]$ *is distributionally concave with* $c'_{\Lambda(M,\psi)} = 1$.

Proof. Let $x_k \in \Lambda(M,\psi)$, $1 \le k \le n$, be arbitrary functions. Set $\alpha_k := \|x_k\|_{\Lambda(M,\psi)}$, $1 \le k \le n$. By the definition of an Orlicz–Lorentz space, we have

$$\int_0^1 M(\alpha_k^{-1} x_k^*) d\psi = 1.$$

Set $y_k := M(\alpha_k^{-1} x_k^*)$, $1 \le k \le n$. Since $M(z^*) = M(z)^*$ for every $z \in S_0[0,1]$, we have $y_k \in \Lambda(\psi)$ and $\|y_k\|_{\Lambda(\psi)} = 1$. Therefore, by Lemma 2.4,

$$\|C(y_1,\ldots,y_n)\|_{\Lambda(\psi)} \ge 1,$$

that is, $\int_0^1 C(y_1,\ldots,y_n)^* d\psi \ge 1$. Since

$$C(y_1,\ldots,y_n) = M(C(\alpha_1^{-1}x_1^*,\ldots,\alpha_n^{-1}x_n^*)),$$

it follows

$$\|C(\alpha_1^{-1}x_1^*,\ldots,\alpha_n^{-1}x_n^*)\|_{\Lambda(M,\psi)} = \int_0^1 M(C(\alpha_1^{-1}x_1^*,\ldots,\alpha_n^{-1}x_n^*)^*)\, d\psi \ge 1.$$

Hence,

$$\|C(x_1,\ldots,x_n)\|_{\Lambda(M,\psi)} \ge \min_{1\le k\le n} \alpha_k. \quad \square$$

2.5. *Interpolation spaces*

Let E, E_0, E_1 be symmetric spaces on I. We say that a symmetric space E is an *interpolation space* between symmetric spaces E_0 and E_1 if $E_0 \cap E_1 \subset E \subset E_0 + E_1$ and every linear operator T which is bounded on E_0 and on E_1 is also bounded on E with $\|T\|_{E\to E} \le C \max_{i=0,1} \|T\|_{E_i \to E_i}$ for some $C > 0$. The set of all interpolation spaces between E_0 and E_1 will be denoted by $Int(E_0, E_1)$.

Given an Orlicz function Φ, we say that Φ is p-convex if the map $t \mapsto \Phi(t^{1/p})$ is convex and q-concave if the map $t \mapsto \Phi(t^{1/q})$ is concave. By convention, every Orlicz function is ∞-concave. The proof of Theorem 4.4 requires the following two results.

Lemma 2.6 (*[32, Lemma 20]*). *Suppose that $M : [0,\infty) \to [0,\infty)$ is such that there exist $1 \le p < q \le \infty$ and a constant $C > 0$ such that for all $0 < s < 1$ we have*

$$C^{-1} s^q M(t) \le M(st) \le C s^p M(t) \quad (t > 0)$$

(where we suppose that the left inequality is missing if $q = \infty$). Then there is an increasing, p-convex and q-concave function M_1 such that with some constant $C_1 > 0$ we have that $C_1^{-1} M(t) \le M_1(t) \le C_1 M(t) (t > 0)$.

Theorem 2.7 (*[17, Theorem 7.1]*). *Let $1 \le p < q \le \infty$ and let $E \in Int(L_p, L_q)$. Then, there is a constant $\lambda_0 > 1$ such that whenever $\|x\|_{L_Q} \le \|y\|_{L_Q}$ for every increasing, p-convex and q-concave function $Q : [0,\infty) \to [0,\infty)$ and $y \in E$, then $x \in E$ with $\|x\|_E \le \lambda_0 \|y\|_E$.*

We also need the definition of the Calderón–Lozanovskiĭ spaces.

Recall that if $\overline{X} = (X_0, X_1)$ is a Banach couple of lattices on I and $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a non-decreasing in each variable and positively homogeneous function (that is, $\phi(\lambda s, \lambda t) = \lambda \phi(s,t)$ for all $\lambda, s, t \ge 0$) with $\phi(0,0) = 0$, then the Calderón–Lozanovskiĭ space $\phi(\overline{X}) := \phi(X_0, X_1)$ consists of all $x \in S_0(I)$ such that $|x| \le \lambda\phi(|x_0|, |x_1|)m$-a.e on I for some $x_j \in X_j$ with $\|x_j\|_{X_j} \le 1$, $j = 0, 1$. The space $\phi(\overline{X})$ is a Banach lattice equipped with the norm (see [26,27])

$$\|x\|_{\phi(\overline{X})} := \inf\left\{\lambda > 0 : \ |x| \le \lambda\phi(|x_0|, |x_1|), \ \|x_0\|_{X_0} \le 1, \|x_1\|_{X_1} \le 1\right\}.$$

In the case of the power function $\phi(s,t) = s^{1-\theta}t^{\theta}$ with $0 < \theta < 1$, the space $\phi(\overline{X})$ is the well-known Calderón space denoted by $X_0^{1-\theta}X_1^{\theta}$ (see [12]). Also note that $\mathcal{F}(\cdot) := \phi(\cdot)$ is an exact positive interpolation functor, that is for a Banach couple $\overline{X} = (X_0, X_1)$ of lattices on I, every positive linear operator T which is bounded on X_0 and on X_1 is also bounded on $\phi(\overline{X})$ with $\|T\|_{\phi(\overline{X})\to\phi(\overline{X})} \le \max_{i=0,1} \|T\|_{X_i \to X_i}$ (see [37]).

2.6. *p-convex and q-concave Banach lattices*

A Banach lattice X is said to be *p-convex* ($1 \le p \le \infty$), respectively, *q-concave* ($1 \le q \le \infty$) if

$$\left\| \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\| \le C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p},$$

respectively,

$$\left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \le C \left\| \left(\sum_{k=1}^{n} |x_k|^q \right)^{1/q} \right\|$$

(with a natural modification in the case $p = \infty$ or $q = \infty$) for some constant $C > 0$ and every choice of vectors $x_1, x_2, \dots, x_n$ in X.

Of course, every Banach lattice is 1-convex and ∞-concave with constant 1. The spaces L^p are p-convex and p-concave with constant 1. If the above estimates hold for pairwise disjoint elements $\{x_k\}_{k=1}^n$ in X, that is,

$$\left\| \sum_{k=1}^{n} x_k \right\| \le C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p},$$

respectively,

$$\left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \le C \left\| \sum_{k=1}^{n} x_k \right\|,$$

then we say that X satisfies an *upper p-estimate* and a *lower q-estimate*, respectively. It is obvious that a p-convex (q-concave) Banach lattice satisfies an upper p-estimate (lower q-estimate).

2.7. *Exchangeable random variables*

Definition 2.8. A finite sequence $\{f_i\}_{i=1}^n$ of random variables (functions) over a probability space is called *exchangeable* (respectively, *symmetrically exchangeable*) if $\{f_{\pi(i)}\}_{i=1}^n$ (respectively, $\{\epsilon_i f_{\pi(i)}\}_{i=1}^n$) have the same joint distribution as $\{f_i\}_{i=1}^n$ for every permutation π of $\{1, 2, \dots, n\}$ (and for every choice of signs $\epsilon_i = \pm 1$).

Let $\mathbf{x} := \{x_i\}_{i=1}^n$ be a finite sequence in a symmetric space $E = E[0,1]$. Let Π_n be the set of all permutations of $\{1,\ldots,n\}$ and let

$$\{I_{\pi,\epsilon} : \pi \in \Pi_n, \epsilon := \{\epsilon_i\}_{i=1}^n \in \{1,-1\}^n\}$$

be a partition of [0, 1] into mutually disjoint intervals, each of length $1/2^n n!$. Let $\psi_{\pi,\epsilon} : I_{\pi,\epsilon} \to [0,1]$ be the unique linear, increasing, onto map. For $1 \le i \le n$ define $s(\mathbf{x})_i \in E$ by

$$s(\mathbf{x})_i(t) := \epsilon_i x_{\pi(i)}(\psi_{\pi,\epsilon}(t)) \quad \text{for } t \in I_{\pi,\epsilon}.$$

Then $\{s(\mathbf{x})_i\}_{i=1}^n$ is a symmetrically exchangeable sequence. In particular, each $s(\mathbf{x})_i (i = 1, 2, \ldots, n)$ has the same distribution and $s(\mathbf{x}) = \{s(\mathbf{x})_i\}_{i=1}^n$ is a 1-symmetric sequence in E. Recall that a sequence $\{y_i\}_{i=1}^n$ from a Banach space X is called K-*symmetric* in X if the inequality

$$K^{-1}\left\|\sum_{i=1}^n a_i y_i\right\|_X \le \left\|\sum_{i=1}^n \epsilon_i a_{\pi(i)} y_i\right\|_X \le K\left\|\sum_{i=1}^n a_i y_i\right\|_X$$

holds for any $\pi \in \Pi_n$, $\{\epsilon_i\}_{i=1}^n \in \{1,-1\}^n$ and $a_i \in \mathbb{R}$.

2.8. The Classification Formula and Theorem

The proofs of our main results appeal to two cornerstone theorems in the theory of isomorphisms of symmetric spaces from the book [15] which we restate below for convenience.

Theorem 2.9 *([15, Theorem 2.1]).* *For every $K \ge 1, C \ge 1$ and any $m \in \mathbb{N}$ there exists a constant $D = D(K,C,m) > 0$ such that if $\{y_i\}_{i=1}^n$ is a finite K-symmetric normalized basic sequence in a Banach lattice Y which is 2-convex and $2m$-concave with both constants less than C, then for every choice of scalars $(a_i)_{i=1}^n$,*

$$D^{-1}\left\|\sum_{i=1}^n a_i y_i\right\|_Y \le \max\left\{\left(\frac{1}{n!}\sum_{\pi\in\Pi_n}\left\|\max_{1\le i\le n}|a_{\pi(i)}y_i|\right\|^{2m}\right)^{1/(2m)},\right.$$
$$\left.\left\|\sum_{i=1}^n y_i\right\|_Y \cdot \left(\frac{1}{n}\sum_{i=1}^n |a_i|^2\right)^{1/2}\right\} \le D\left\|\sum_{i=1}^n a_i y_i\right\|_Y.$$

It is important to emphasize that the left-hand side inequality from Theorem 2.9 holds in every q-concave $(q < \infty)$ Banach lattice Y (see [15, Remark 1, p. 63]).

Theorem 2.10 *([15, Theorem 6.1]).* *Let X be a symmetric space on $[0,1]$ such that the Haar system is an unconditional basis for X. Let Y be a symmetric space on $[0,1]$ or $[0,\infty)$ which does not contain uniformly isomorphic copies of l_∞^n for all $n \in \mathbb{N}$. If X embeds isomorphically into Y then one of the following three (non-exclusive) possibilities holds.*

(1) *There is a constant $C > 0$ such that*

$$\|f\|_Y \le C\|f\|_X$$

for all $f \in X$.

(2) *The Haar system in X is equivalent to a sequence of disjointly supported functions in Y.*

(3) *X is isomorphic to $L_2[0,1]$.*

2.9. Scales

A scale is an indexed set of Banach spaces $\{X_\alpha\}_{\alpha\in\mathcal{A}}$. We assume that the index family set $\mathcal{A}$ has been (partially) ordered as follows

$$\alpha \le \beta \Leftrightarrow X_\beta \subset^1 X_\alpha, \quad \alpha, \beta \in \mathcal{A}$$

($X \subset^1 Y$ means that X is contained in Y with embedding constant 1). A scale $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be *compatible* if there exists a Hausdorff topological vector space $\mathcal{U}$ such that each X_α is algebraically and topologically embedded in $\mathcal{U}$. A scale $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is *strongly compatible* if there exists a Banach space $\tilde{X}$ such that $X_\alpha \subset^1 \tilde{X}$, $\alpha \in \mathcal{A}$. Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a compatible scale, $\|x\|_\alpha := \|x\|_{X_\alpha}$. Let

$$\Delta(X_\alpha)_{\alpha\in\mathcal{A}} := \left\{ x \in \bigcap_{\alpha\in\mathcal{A}} X_\alpha : \ \|x\|_{\Delta(X_\alpha)} := \sup_{\alpha\in\mathcal{A}} \|x\|_\alpha < \infty \right\}.$$

Then $(\Delta(X_\alpha)_{\alpha\in\mathcal{A}}, \|\cdot\|_{\Delta(X_\alpha)})$ is a Banach space with the following properties:

(i) $\Delta(X_\alpha)_{\alpha\in\mathcal{A}} \subset^1 X_\alpha$, $\forall \alpha \in \mathcal{A}$;

(ii) if F is a Banach space such that $F \subset^1 X_\alpha$, $\forall \alpha \in \mathcal{A}$, then $F \subset^1 \Delta(X_\alpha)_{\alpha\in\mathcal{A}}$.

A scale $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be *total*, if

$$\Delta(X_\alpha)_{\alpha\in\mathcal{A}} = \bigcap_{\alpha\in\mathcal{A}} X_\alpha$$

as linear spaces.

Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a scale and let

$$U(X_\alpha)_{\alpha\in\mathcal{A}} := \{x : \ x \in X_\alpha, \ \text{for some } \alpha \in \mathcal{A}\}.$$

There exists a natural homogeneous functional that can be defined on $U(X_\alpha)_{\alpha\in\mathcal{A}}$:

$$\|x\|_{U(X_\alpha)} := \inf_{\alpha\in\mathcal{A}} \|x\|_\alpha.$$

If $(U(X_\alpha)_{\alpha\in\mathcal{A}}, \|\cdot\|_{U(X_\alpha)})$ is a Banach space, then the scale $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be *U-complete*.

Suppose that $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a strongly compatible scale. Let

$$\Sigma(X_\alpha)_{\alpha\in\mathcal{A}} := \left\{ \sum_{\alpha\in\mathcal{A}} x_\alpha, \ \text{where } x_\alpha \in X_\alpha \text{ for every } \alpha \in \mathcal{A} \right\},$$

and convergence is understood as an absolute convergence in $\tilde{X}$. We endow $\Sigma(X_\alpha)_{\alpha\in\mathcal{A}}$ with the norm

$$\|x\|_{\Sigma(X_\alpha)} := \inf \left\{ \sum_{\alpha\in\mathcal{A}} \|x_\alpha\|_\alpha : \ x = \sum_{\alpha\in\mathcal{A}} x_\alpha \right\},$$

where the infimum is taken over all absolutely convergent in $\tilde{X}$ series $x = \sum_{\alpha\in\mathcal{A}} x_\alpha$ with $x_\alpha \in X_\alpha$ for every $\alpha \in \mathcal{A}$. Then $(\Sigma(X_\alpha)_{\alpha\in\mathcal{A}}, \|\cdot\|_{\Sigma(X_\alpha)})$ is the smallest Banach space with the property $X_\beta \subset^1 \Sigma(X_\alpha)_{\alpha\in\mathcal{A}}$, $\forall \beta \in \mathcal{A}$.

Theorem 2.11 *([30, Theorem 1]).* *Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a strongly compatible scale. Then the following conditions are equivalent:*

(i) *the scale $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is U-complete;*

(ii) $U(X_\alpha)_{\alpha\in\mathcal{A}} = \Sigma(X_\alpha)_{\alpha\in\mathcal{A}}$;
(iii) *for every sequence* $\{\alpha_n\}_{n\geq 1}\subset\mathcal{A}$, $\Sigma(X_{\alpha_n})_{n\geq 1}\subset U(X_\alpha)_{\alpha\in\mathcal{A}}$.

3. Proofs of main results

This section contains the proofs of Theorems 1.2 and 1.3 and Corollaries 1.4 and 1.5. We begin with the following lemma, which demonstrates the utility of the notion of distributional concavity in the study of symmetric structure.

Lemma 3.1. *Let E be a distributionally concave symmetric space on* $[0,1]$ *which is q-concave for some $q<\infty$. Then for every $K>0$ there exists a constant $C=C(E,K)>0$ such that for every K-symmetric sequence $\boldsymbol{x}=\{x_i\}_{i=1}^n\subset E$ and any $a_i\in\mathbb{R}$ the following inequality holds*

$$\left\|\sum_{i=1}^n a_i x_i\right\|_E \leq C\left\|\sum_{i=1}^n a_i s(\boldsymbol{x})_i\right\|_E,$$

where $s(\boldsymbol{x})=\{s(\boldsymbol{x})_i\}_{i=1}^n$ is a symmetrically exchangeable sequence constructed for the sequence $\boldsymbol{x}$ according to the procedure described in Section 2.7.

Proof. Since E is a q-concave space for $q<\infty$, it follows by [24, Theorem 1.d.6] that there exists $B=B(E)>0$ such that for an arbitrary $\{f_i\}_{i=1}^n\subset E$ the following holds

$$\frac{1}{\sqrt{2}}\left\|\left(\sum_{i=1}^n f_i^2\right)^{1/2}\right\|_E \leq \int_0^1\left\|\sum_{i=1}^n r_i(t)f_i\right\|_E dt \leq B\left\|\left(\sum_{i=1}^n f_i^2\right)^{1/2}\right\|_E, \tag{3.1}$$

where $r_i\,(i=1,2,\ldots)$ is the ith Rademacher function on $[0,1]$ (see e.g. [24]).

Let us recall (see Section 2.7) that the collection of pairwise disjoint intervals (each of length $1/2^n n!$)

$$\{I_{\pi,\epsilon}\ :\ \pi\in\Pi_n,\ \epsilon:=\{\epsilon_i\}_{i=1}^n\in\{1,-1\}^n\}$$

forms a partition of $[0,1]$ and that $\psi_{\pi,\epsilon}:I_{\pi,\epsilon}\to[0,1]$ is the unique linear, increasing, onto map. For every permutation $\pi\in\Pi_n$ we define the set

$$\mathcal{I}_\pi := \bigcup_{\epsilon\in\{1,-1\}^n} I_{\pi,\epsilon}$$

and the mapping

$$\varphi_\pi:\mathcal{I}_\pi\to[0,1],\quad \varphi_\pi(x)=\psi_{\pi,\epsilon}(x)\quad \text{if } x\in I_{\pi,\epsilon}\ (\epsilon\in\{1,-1\}^n).$$

Since $m(\mathcal{I}_\pi)=1/n!$ and $m(\varphi_\pi^{-1}(A))=m(A)/n!$ for every measurable $A\subset[0,1]$, it follows that $d_{f\circ\varphi_\pi}=d_f/n!$ for any measurable function $f:[0,1]\to\mathbb{R}$.

For every permutation $\pi\in\Pi_n$ define a linear operator $\mathcal{D}_\pi$ on $L_0[0,1]$ by setting

$$\mathcal{D}_\pi f(x)=(f\circ\varphi_\pi)(x)\quad \text{if } x\in\mathcal{I}_\pi \text{ and } \mathcal{D}_\pi f(x)=0 \text{ if } x\notin\mathcal{I}_\pi.$$

It follows that for every measurable function f, the functions $\mathcal{D}_\pi f$ and $\sigma_{\frac{1}{n!}}f$ are equimeasurable. Moreover, by the definition of the sequence $s(\mathbf{x})=\{s(\mathbf{x})_i\}_{i=1}^n$ we have

$$\sum_{i=1}^n (a_i s(\mathbf{x})_i)^2 = \sum_{\pi\in\Pi_n}\mathcal{D}_\pi\left(\sum_{i=1}^n (a_i x_{\pi(i)})^2\right). \tag{3.2}$$

Indeed, for a particular choice of $t \in (0,1)$, the external sum on the right hand side of (3.2) reduces to a single summand.

Since the sequence $s(\mathbf{x})$ is symmetrically exchangeable, using (3.1) and (3.2) we obtain

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i s(\mathbf{x})_i\right\|_E &\geq \frac{1}{\sqrt{2}}\left\|\left(\sum_{i=1}^{n}(a_i s(\mathbf{x})_i)^2\right)^{1/2}\right\|_E\\ &= 2^{-1/2}\left\|\left(\sum_{\pi\in\Pi_n}\mathcal{D}_\pi\left(\sum_{i=1}^{n}(a_i x_{\pi(i)})^2\right)\right)^{1/2}\right\|_E\\ &= 2^{-1/2}\left\|\sum_{\pi\in\Pi_n}\mathcal{D}_\pi\left(\sum_{i=1}^{n}(a_i x_{\pi(i)})^2\right)^{1/2}\right\|_E.\end{aligned}$$

Let us point out that the functions

$$\mathcal{D}_\pi\left(\sum_{i=1}^{n}(a_i x_{\pi(i)})^2\right)^{1/2}, \quad \pi \in \Pi_n$$

are pairwise disjointly supported and equimeasurable with the functions

$$\sigma_{\frac{1}{n!}}\left(\sum_{i=1}^{n}(a_i x_{\pi(i)})^2\right)^{1/2}, \quad \pi \in \Pi_n,$$

respectively. Using this observation and distributional concavity of E, we arrive at

$$\left\|\sum_{i=1}^{n} a_i s(\mathbf{x})_i\right\|_E \geq \frac{1}{\sqrt{2}\cdot c'_E}\min_{\pi\in\Pi_n}\left\|\left(\sum_{i=1}^{n}(a_i x_{\pi(i)})^2\right)^{1/2}\right\|_E,$$

where c'_E is the constant from Definition 2.2.

Next, using K-symmetricity of $\{x_i\}_{i=1}^n$ and (3.1) we obtain

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i s(\mathbf{x})_i\right\|_E &\geq \frac{1}{\sqrt{2}K\cdot c'_E B}\min_{\pi\in\Pi_n}\left\|\sum_{i=1}^{n} a_i x_{\pi(i)}\right\|_E\\ &\geq \frac{1}{\sqrt{2}K^2\cdot c'_E B}\left\|\sum_{i=1}^{n} a_i x_i\right\|_E,\end{aligned}$$

which proves the lemma. □

Remark 3.2. It follows from the definition of the operator $\mathcal{D}_\pi, \pi \in \Pi_n$ that, for any $f \in L_0[0,1]$, the function $\sum_{\pi\in\Pi_n}\mathcal{D}_\pi f$ is equimeasurable with f itself. Therefore, if $\{x_i\}_{i=1}^n$ and $\{s(\mathbf{x})_i\}_{i=1}^n$ be as in Lemma 3.1, the functions

$$\left(\sum_{\pi\in\Pi_n}\mathcal{D}_\pi\left(\sum_{i=1}^{n} x_{\pi(i)}^2\right)\right)^{1/2} \quad \text{and} \quad \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

are equimeasurable. Thus, from (3.2) it follows that

$$\left\|\left(\sum_{i=1}^n s(\mathbf{x})_i^2\right)^{1/2}\right\|_E=\left\|\left(\sum_{i=1}^n x_i^2\right)^{1/2}\right\|_E.$$

Moreover, if E is a q-concave space ($q<\infty$), then, by (3.1), we have

$$\begin{aligned}\Big\|\sum_{i=1}^n s(\mathbf{x})_i\Big\|_E &\le B\cdot\left\|\left(\sum_{i=1}^n s(\mathbf{x})_i^2\right)^{1/2}\right\|_E\\ &= B\cdot\left\|\left(\sum_{i=1}^n x_i^2\right)^{1/2}\right\|_E\\ &\le \sqrt{2}B\cdot K\left\|\sum_{i=1}^n x_i\right\|_E.\end{aligned}\tag{3.3}$$

The following lemma links tensor product operators acting on the products $E\times E$ and $E\times Z_E^2$.

Lemma 3.3. *Let E be a symmetric function space on $[0,1]$ such that E is an interpolation space for the couple (L_2,L_∞). Suppose that the tensor product operator maps $E\times E$ into E. Then*

(i) *the tensor product operator maps $E\times Z_E^2$ into Z_E^2;*
(ii) *there exists $C>0$ such that for every $n\in\mathbb{N}$, any $a_n\in\mathbb{R}$ and an arbitrary sequence of pairwise disjoint equimeasurable functions $\{f_k\}_{k=1}^n\subset Z_E^2=Z_E^2[0,\infty)$ the following inequality holds*

$$\left\|\sum_{k=1}^n a_kf_k\right\|_{Z_E^2}\le C\left\|\sum_{k=1}^n f_k\right\|_{Z_E^2}\cdot\left\|\sum_{k=1}^n a_ke_{n,k}\right\|_E,$$

where

$$e_{n,k}=\chi_{[\frac{k-1}{n},\frac{k}{n})},\quad n\in\mathbb{N},\ k=1,2,\dots,n.$$

Proof. (i) Without loss of generality, we assume that $x=x^*\in E$ and $y=y^*\in Z_E^2$. Denote $y_1:=y\chi_{[0,1]}$, $y_2:=y\chi_{[1,\infty)}$. By the definition of the space Z_E^2, we have $y_1\in E$ and $y_2\in L_2[0,\infty)$. By the assumption, we know that $x\otimes y_1\in E$, and so $x\otimes y_1\in Z_E^2$.

Next, denote $z_1:=(x\otimes y_2)^*\chi_{[0,1]}$, $z_2:=(x\otimes y_2)^*\chi_{[1,\infty)}$. Since $E\in Int(L_2,L_\infty)$, it follows that $E\subset L_2$. Hence, $x\in L_2$ and $x\otimes y_2\in L_2[0,\infty)$. Consequently, $z_2\in(L_\infty\cap L_2)[0,\infty)$ and, therefore, $z_2\in Z_E^2$.

So, we only need to check that $z_1\in E$. For a fixed $z\in(L_\infty\cap L_2)[0,\infty)$ let us consider the following sublinear operator on $L_2[0,1]$:

$$(T_zx)(t):=\frac{1}{t}\int_0^t(x\otimes z)^*(s)\,ds\cdot\chi_{[0,1]}(t),\quad 0<t\le 1.$$

It is easy to see that T_z is bounded on both $L_2[0,1]$ and $L_\infty[0,1]$. According to [25], the interpolation in the couple (L_2,L_∞) is described by the real method of interpolation (see e.g. [10] for the definition), so the operator T_z is bounded in E as well.

Consequently, if $x \in E$ and $z \in (L_\infty \cap L_2)[0,\infty)$, then $(x \otimes z)^* \chi_{[0,1]} \in E$. Finally, since $y_2 := y\chi_{[1,\infty)} \in (L_\infty \cap L_2)[0,\infty)$, we conclude that $z_1 = (x \otimes y_2)^* \chi_{[0,1]} \in E$, which proves the first assertion.

(ii) Denote $x := \sum_{k=1}^n a_k e_{n,k} \in E$ and $y := \sum_{k=1}^n f_k \in Z_E^2$. Since all functions f_k are pairwise disjoint and equimeasurable, for every $\tau > 0$ we have

$$(m \otimes m)(\{|x \otimes y| > \tau\}) = \frac{1}{n} \sum_{k=1}^n \sum_{i=1}^n m\left(\left\{|f_i| > \frac{\tau}{|a_k|}\right\}\right) = \sum_{k=1}^n m\left(\left\{|f_1| > \frac{\tau}{|a_k|}\right\}\right) = m\left(\left\{\left|\sum_{k=1}^n a_k f_k\right| > \tau\right\}\right),$$

where $m \otimes m$ is the 2-dimensional Lebesgue measure.

So, we have shown that functions $\sum_{k=1}^n a_k f_k$ and $x \otimes y$ are equimeasurable. The assertion follows now from (i). □

Lemma 3.4. *Let E be a symmetric space on $[0,1]$ equipped with an order semicontinuous norm, such that $E \in Int(L_2, L_\infty)$. There exists $C > 0$, such that for every symmetrically exchangeable sequence $\{y_i\}_{i=1}^n \in E$ and any $a_i \in \mathbb{R}$ the following holds*

$$\left\|\sum_{i=1}^n a_i \overline{y}_i\right\|_{Z_E^2} \le C \cdot \left\|\sum_{i=1}^n a_i y_i\right\|_E .$$

Here, the sequence $\{\overline{y}_i\}$ consists of pairwise disjoint equimeasurable copies of y_i ($i = 1, 2, \dots, n$).

Proof. Define the operator $A : E(l_2) \to Z_E^2$ as follows

$$A(\{x_i\}_{i=1}^\infty) := \sum_{i=1}^\infty x(t - i + 1)\chi_{[i-1,i)}(t), \quad t > 0.$$

Since $E \in Int(L_2, L_\infty)$, it follows that A is a bounded operator [9, proof of Corollary 3.6]. Hence, there exists $C_1 > 0$, such that

$$\left\|\sum_{i=1}^n a_i \overline{y}_i\right\|_{Z_E^2} \le C_1 \cdot \left\|\left(\sum_{i=1}^n a_i^2 y_i^2\right)^{1/2}\right\|_E . \tag{3.4}$$

It follows from [24, Theorem 1.d.6] that the left inequality in (3.1) holds in every symmetric space. Hence, using (3.1) and the symmetric exchangeability of $\{y_i\}_{i=1}^n$, we obtain

$$\left\|\left(\sum_{i=1}^n a_i^2 y_i^2\right)^{1/2}\right\|_E \le \sqrt{2} \left\|\sum_{i=1}^n a_i y_i\right\|_E .$$

The required result follows from (3.4) and the preceding inequality. □

Lemma 3.4 allows us to extend Lemma 3.3 from the case of pairwise disjoint equimeasurable functions in Z_E^2 to the case of an arbitrary symmetrically exchangeable sequence in E.

Lemma 3.5. *Let $q < \infty$. Let E be a q-concave symmetric space on $[0,1]$ equipped with an order semicontinuous norm, such that $E \in Int(L_2, L_\infty)$. If the tensor product operator is bounded* *from $E \times E$ into E, then there exists $C > 0$, such that for every symmetrically exchangeable sequence $\{y_k\}_{k=1}^n \in E$ and any $a_k \in \mathbb{R}$ the following holds*

$$\left\|\sum_{k=1}^n a_k y_k\right\|_E \le C \cdot \left\|\sum_{k=1}^n y_k\right\|_E \cdot \left\|\sum_{k=1}^n a_k e_{n,k}\right\|_E .$$

Proof. It follows from the assumption that the sequence $\{y_k\}_{k=1}^n$ consists of equimeasurable functions. Let us consider the sequence $\{\overline{y}_k\}_{k=1}^n$ of its disjoint translates, that is,

$$\bar{y}_k(t) := y_k(t-k+1)\chi_{[k-1,k)}(t), \quad t \in \mathbb{R}.$$

Applying Lemmas 3.3 and 3.4 to the latter sequence, we obtain

$$\begin{aligned} \left\|\sum_{k=1}^n a_k \overline{y}_k\right\|_{Z_E^2} &\le C_1 \cdot \left\|\sum_{k=1}^n \overline{y}_k\right\|_{Z_E^2} \cdot \left\|\sum_{k=1}^n a_k e_{n,k}\right\|_E \\ &\le C_2 \cdot \left\|\sum_{k=1}^n y_k\right\|_E \cdot \left\|\sum_{k=1}^n a_k e_{n,k}\right\|_E . \end{aligned} \tag{3.5}$$

Recall that the left inequality in Theorem 2.9 holds in every q-concave ($q < \infty$) Banach lattice (see Section 2.6). So, using that inequality and the fact that $\{y_k\}_{k=1}^n$ is a symmetrically exchangeable sequence, we obtain

$$\left\|\sum_{k=1}^n a_k y_k\right\|_E \le C_3 \max\left\{\left\|\max_{i=1,\ldots,n} |a_i y_i|\right\|_E, \left\|\sum_{k=1}^n y_k\right\|_E \cdot \left(\frac{1}{n}\sum_{k=1}^n a_k^2\right)^{1/2}\right\}.$$

Since

$$(\max_{i=1,\ldots,n} |a_i y_i|)^* \le \left(\sum_{i=1}^n a_i \overline{y}_i\right)^*,$$

it follows from (3.5) that

$$\begin{aligned} \left\|\sum_{k=1}^n a_k y_k\right\|_E &\le C_3 \max\left\{\left\|\sum_{k=1}^n a_k \overline{y}_k\right\|_{Z_E^2}, \left\|\sum_{k=1}^n y_k\right\|_E \cdot \left(\frac{1}{n}\sum_{k=1}^n a_k^2\right)^{1/2}\right\} \\ &\le C_2 C_3 \cdot \left\|\sum_{k=1}^n y_k\right\|_E \cdot \max\left\{\left\|\sum_{k=1}^n a_k e_{n,k}\right\|_E, \left\|\sum_{k=1}^n a_k e_{n,k}\right\|_{L_2}\right\} \\ &= C_2 C_3 \cdot \left\|\sum_{k=1}^n y_k\right\|_E \cdot \left\|\sum_{k=1}^n a_k e_{n,k}\right\|_E , \end{aligned}$$

where the last inequality holds since $E \in Int(L_2, L_\infty)$ and so $E \subset L_2$ (without loss of generality we can assume that $\|x\|_E \ge \|x\|_{L_2}$ for every $x \in E$). □

Now, we are fully prepared to prove our first major result in this paper.

Proof of Theorem 1.2. We first show that the conditions of Lemmas 3.1, 3.3 and 3.4 and Classification Theorem 2.10 (see Section 2.8) hold.

It is given that $E \neq L_\infty$ and that the operator $(x, y) \to x \otimes y$ is bounded from $E \times E$ to E. Hence, applying [4, Corollary 1.2] (see also [5, Corollary 2]), we obtain that the Boyd index $q_E < \infty$.

Since $E \in Int(L_2, L_\infty)$, we conclude by [7, Remark 2] that $E \in Int(L_2, L_q)$ for every $q > q_E$. By Theorem 4.6 below, we derive that E is q-concave. Hence, by [24, Proposition 1.f.3(i)], the space E is of cotype q and therefore it does not contain uniformly isomorphic copies of l_∞^n, $n \in \mathbb{N}$.

Moreover, since E is a separable symmetric space such that $E \in Int(L_2, L_q)$, it follows that the Haar system forms an unconditional basis in E (see e.g. [22,24]). Hence, if a symmetric space F on $[0, 1]$ is isomorphic to some subspace of E, then the Haar system forms an unconditional basis in F as well [24, Corollary 2.c.11].

Therefore we can apply Theorem 2.10 and derive that either $F = L_2[0, 1]$ or the Haar system in F is equivalent to a sequence of disjointly supported functions in E or for some $C > 0$ and all $f \in F$ we have

$$\|f\|_E \le C\|f\|_F. \tag{3.6}$$

The first two options coincide with alternatives given in the statement of Theorem 1.2. Thus, we only need to show that if the third option above holds, then necessarily $F = E$ (up to the norm equivalence).

Let $T : F \to E$ be an isomorphism. Denote

$$x_{n,i} := Te_{n,i},$$

where $e_{n,i} = \chi_{[(i-1)/n, i/n)}$, $n \in \mathbb{N}$, $i = 1, \dots, n$.

If $K := \|T\| \cdot \|T^{-1}\|$, then for every $n \in \mathbb{N}$ the sequence $\mathbf{x}_n = \{x_{n,i}\}_{i=1}^n$ is K-symmetric in E. Hence, by Lemmas 3.1 and 3.5, for every $a_i \in \mathbb{R}$, we have

$$\left\| \sum_{i=1}^n a_i x_{n,i} \right\|_E \le C_1 \left\| \sum_{i=1}^n s(\mathbf{x}_n)_i \right\|_E \cdot \left\| \sum_{i=1}^n a_i e_{n,i} \right\|_E, \tag{3.7}$$

where $s(\mathbf{x}_n) = \{s(\mathbf{x}_n)_i\}_{i=1}^n$ is the symmetrically exchangeable sequence constructed for the sequence $\mathbf{x}_n = \{x_{n,i}\}_{i=1}^n$, according to the procedure described in Section 2.7.

Without loss of generality, assume that $\|\chi_{[0,1]}\|_E = 1$. Then by (3.3) we obtain

$$\left\| \sum_{i=1}^n s(\mathbf{x}_n)_i \right\|_E \le C_2 \left\| \sum_{i=1}^n x_{n,i} \right\|_E = C_2 \left\| T\left(\sum_{i=1}^n e_{n,i} \right) \right\|_E \le C_2\|T\|.$$

Noting that

$$\left\| \sum_{i=1}^n a_i e_{n,i} \right\|_F \le \|T^{-1}\| \cdot \left\| \sum_{i=1}^n a_i x_{n,i} \right\|_E,$$

we infer from (3.7) that

$$\left\| \sum_{i=1}^n a_i e_{n,i} \right\|_F \le C_3 \left\| \sum_{i=1}^n a_i e_{n,i} \right\|_E,$$

where $C_3 > 0$ independent of $n \in \mathbb{N}$, $i = 1, \dots, n$ and $a_i \in \mathbb{R}$.

Finally, since the set

$$\left\{ \sum_{i=1}^{n} a_i e_{n,i} \ : \ n \in \mathbb{N}, a_i \in \mathbb{R} \right\}$$

is dense in E, we obtain that $\|f\|_F \le C_3 \|f\|_E$ for every $f \in F$. The latter estimate together with (3.6) shows that $E = F$. This completes the proof of Theorem 1.2. □

In the proof of Theorem 1.3, we shall employ the following notion.

Definition 3.6. Let $p \ge 1$. A Banach space X is said to have the p-Banach–Saks-property if every weakly convergent to zero sequence $\{x_n\} \subset X$ contains a subsequence $\{x_{n_k}\}$ such that

$$\sup_{m \in \mathbb{N}} m^{-1/p} \left\| \sum_{k=1}^{m} x_{n_k} \right\|_X < \infty.$$

Remark 3.7. It should be pointed out that if X is a separable symmetric sequence space (see e.g. [23]), or just a separable Banach lattice of bounded sequences, then it has the p-Banach–Saks-property whenever X satisfies the upper p-estimate (see e.g. [24]). Indeed, it is well-known (and easy to show, see e.g. the beginning of Section 4 in [8]), that in this case, for every weakly null sequence $\{x_n\} \subset X$, there exist a subsequence $\{x_{n_k}\}$ and sequence $\{y_k\} \subset X$ of pairwise disjoint elements, such that

$$\|x_{n_k} - y_k\|_X < 2^{-k}, \quad k \in \mathbb{N}.$$

Proof of Theorem 1.3. (2) $\Rightarrow$ (1). Reasoning like in the proof of Theorem 1.2, we deduce that $q_E < \infty$. Moreover, since E satisfies an upper p-estimate for $p > 2$, it follows that $p_E \ge p > 2$ (see e.g. [24, p. 132]). Consequently, by the Boyd Interpolation Theorem (see e.g. [24, Theorem 2.b.11]), we have $E \in Int(L_2, L_q)$ for all $q > q_E$. Therefore, we may conclude that the assumptions of Theorem 1.2 hold. Thus, in order to complete the proof of the implication (2) $\Rightarrow$ (1) in Theorem 1.3, it suffices to show that the Haar system in F is not equivalent to a sequence of disjointly supported functions in E.

Suppose, for a contradiction, that the latter option holds. Then F is isomorphic to the closed (in E) linear span $[f_k]_{k=1}^{\infty}$ of a sequence $\{f_k\}_{k=1}^{\infty}$ of pairwise disjoint functions, such that $\|f_k\|_E = 1$ for every $k = 1, 2, \ldots$. In other words, the symmetric space F is isomorphic to the Banach lattice

$$X := \left\{ a = (a_i)_{i=1}^{\infty} \ : \ \|a\|_X := \left\| \sum_{i=1}^{\infty} a_i f_i \right\|_E < \infty \right\}.$$

Let $a^k = (a_i^k)_{i=1}^{\infty} (k = 1, 2, \ldots, m)$ be a sequence of pairwise disjointly supported elements from X. Since E satisfies the upper p-estimate, it follows that for some pairwise disjoint subsets $U_k \subset \mathbb{N}$ we have

$$\left\| \sum_{k=1}^{m} a^k \right\|_X = \left\| \sum_{k=1}^{m} \sum_{i \in U_k} a_i^k f_i \right\|_E \le C \left(\sum_{k=1}^{m} \left\| \sum_{i \in U_k} a_i^k f_i \right\|_E^p \right)^{1/p} = C \left(\sum_{k=1}^{m} \left\| a^k \right\|_X^p \right)^{1/p}.$$

By Definition 3.6 and Remark 3.7, we conclude that X has the p-Banach–Saks-property with $p > 2$. Since F is isomorphic to the space X, we infer that the space F also has the p-Banach–Saks-property with $p > 2$. However, it follows from [36, Lemma 4.1] that a symmetric function space can only have the p-Banach–Saks-property when $p \leq 2$. This contradiction proves the implication (2) $\Rightarrow$ (1).

Let us now prove the implication (1) $\Rightarrow$ (2). Suppose that the tensor product operator is not bounded from $E \times E$ to E. A standard argument then shows that there exists an element $g \in E$ such that $\|g\|_E = 1$ and $g \otimes g \notin E([0,1] \times [0,1])$. Fix such g and following [15, Section 7, p.189], define a new norm on $L_\infty[0,1]$ by setting

$$\|f\|_g := \|f \otimes g\|_{E([0,1]\times[0,1])}.$$

Denote by E_g the completion of $L_\infty[0,1]$ with respect to the new norm. By [15, p. 189], $(E_g, \|\cdot\|_g)$ is a symmetric space on $[0,1]$ and the mapping

$$T_1 : E_g \to E([0,1] \times [0,1])$$

is a lattice isomorphism of E_g onto some sublattice of $E([0,1] \times [0,1])$.

Let now $\omega : [0,1] \times [0,1] \to [0,1]$ be a one-to-one (up to sets of measure zero) measure preserving mapping and $T_\omega f := f(\omega)$. We have that the operator $T = T_\omega T_1$ is an isomorphic embedding of the symmetric space E_g into E.

Since $g \otimes g \notin E([0,1] \times [0,1])$, it follows that $E_g \neq E$. Mimicking the argument from the proof of [14, Proposition 1], we shall prove that $E_g \neq L_2$.

Set, as above, $e_{n,i} = \chi_{[(i-1)/n, i/n)}$ for $n \in \mathbb{N}$ and $i = 1, \ldots, n$. We have $\|e_{n,i}\|_{E_g} = \|e_{n,1}\|_{E_g}$ for every $i = 1, \ldots, n$. Since E satisfies the upper p-estimate, it follows that

$$\begin{aligned}
\left\| \sum_{i=1}^n e_{n,i} \right\|_{E_g} &= \left\| \sum_{i=1}^n e_{n,i} \otimes g \right\|_{E([0,1]\times[0,1])} \\
&\leq C \left(\sum_{i=1}^n \|e_{n,i} \otimes g\|^p_{E([0,1]\times[0,1])} \right)^{1/p} \\
&= C \cdot n^{1/p} \|e_{n,1}\|_{E_g}.
\end{aligned}$$

Consequently, for every $n \in \mathbb{N}$ we have

$$\left\| \sum_{i=1}^n e_{n,i} \right\|_{E_g} \cdot \left(\sum_{i=1}^n \|e_{n,i}\|^2_{E_g} \right)^{-\frac{1}{2}} \leq C \cdot n^{\frac{1}{p} - \frac{1}{2}}.$$

Since $p > 2$, it follows that the left-hand side of the latter inequality tends to zero as $n \to \infty$. So, $E_g \neq L_2$.

Summing up, we have a symmetric space E_g on $[0,1]$ which is isomorphic to a subspace of E and $E_g \neq E$, $E_g \neq L_2$. This contradicts to the condition (1) of Theorem 1.3. $\square$

Corollary 1.4 follows immediately from Theorem 1.3. We complete this section by proving Corollary 1.5.

Proof of Corollary 1.5. Let functions M and ψ satisfy the assumptions of Corollary 1.5. In this case, it is sufficient to show that the Orlicz–Lorentz space $\Lambda(M, \psi)$ satisfies assumptions of Theorem 1.3.

First, let us observe that $N(t) := M(t^{1/p})$ is a convex increasing function on $[0, \infty)$. Using the definition of an Orlicz–Lorentz norm, we obtain

$$\left\| \left(\sum_{k=1}^{m} x_k^p \right)^{1/p} \right\|_{\Lambda(M,\psi)} = \left\| \sum_{k=1}^{m} x_k^p \right\|_{\Lambda(N,\psi)}^{1/p} \leq \left(\sum_{k=1}^{m} \| x_k^p \|_{\Lambda(N,\psi)} \right)^{1/p} = \left(\sum_{k=1}^{m} \| x_k \|_{\Lambda(M,\psi)}^p \right)^{1/p}$$

for every $x_k \in \Lambda(M, \psi)$, $x_k \geq 0$ $(k = 1, \ldots, m)$. Hence, the space $\Lambda(M, \psi)$ is p-convex. In particular, it satisfies an upper p-estimate. Moreover, by (1.1), the function M satisfies the Δ_2-condition at infinity. Hence, as in the case of an Orlicz space, we conclude that $\Lambda(M, \psi)$ is a separable space (see also [18]).

By Proposition 2.5, we have that $\Lambda(M, \psi)$ is a distributionally concave space. So, now we only need to show that the tensor product operator is bounded from $\Lambda(M, \psi) \times \Lambda(M, \psi)$ into $\Lambda(M, \psi)$.

Denote by M^{-1} a continuous function, such that $M \circ M^{-1} = Id$. Let

$$\phi(s, t) = t M^{-1} \left(\frac{s}{t} \right), \quad \text{for } t > 0, s > 0$$

and $\phi(s, 0) = 0$.

It is easy to check that

$$\Lambda(M, \psi) = \phi(\Lambda(\psi), L_\infty), \tag{3.8}$$

where $\Lambda(\psi)$ is the Lorentz space and $\phi(\Lambda(\psi), L_\infty)$ is the Calderón–Lozanovskiĭ space, constructed by the couple $(\Lambda(\psi), L_\infty)$ (see Preliminaries).

Since ψ satisfies inequality (1.2), it follows from [31, Theorem B] (see also [2]) that the tensor product operator $x \otimes y$ is bounded from $\Lambda(\psi) \times \Lambda(\psi)$ to $\Lambda(\psi)$. Clearly, it is also bounded from $L_\infty \times L_\infty$ to L_∞.

Let us mention that, by (1.1), the function $\phi(1, t) = t M^{-1}(\frac{1}{t})$ is supermultiplicative for $0 < t \leq 1$. Hence, the functor $\phi(\cdot, \cdot)$ interpolates bilinear operators, in particular, $\phi(\cdot, \cdot)$ interpolates the tensor product operator (see [3, Theorem 7] or [29]). Hence, it follows from (3.8) that the tensor product operator is bounded from $\Lambda(M, \psi) \times \Lambda(M, \psi)$ into $\Lambda(M, \psi)$. □

4. Distributional concavity and Orlicz spaces

In the proof of Theorem 1.2 we used the fact that every distributionally concave symmetric space E such that $E \in Int(L_2, L_q)$ is q-concave (see below Theorem 4.6). In this section we consider properties of distributionally concave symmetric spaces in detail.

In the proof of the main theorem of this section, we will use the well-known version of the Hahn–Banach Theorem about the separation of convex sets, one of which is compact and the other is closed, in locally convex spaces (see, for example, [11, II, Section 3]). We will apply this theorem to the dual space (equipped with the weak*-topology) of the space $(C_0[0, L], \|\cdot\|_C)$ of all continuous on the segment $[0, L]$ functions which are 0 at 0. The classical Riesz representation theorem (see e.g. [20], VI, Section 6, Theorem 4′) states that if $(C[0, L], \|\cdot\|_C)$ is the space of all continuous functions on $[0, L]$, then its dual space is isometric to the space $V[0, L]$, which consists of all right-continuous everywhere on $[0, L]$, except possibly 0, functions f, of bounded variation with $f(L) = 0$. The norm in this space is given by the variation of a function on $[0, L]$.

Recall that if Y is a closed subspace of a Banach space X, then we can identify the dual space Y^* with the quotient $X^*/A(Y)$ where $A(Y)$ is the annihilator of Y. It is clear that the annihilator of the subspace $C_0[0, L]$ in the space $C[0, L]$ is the one-dimensional subspace of the space $V[0, L]$ generated by the functional $\phi(f) = f(0)$. Therefore, the dual space to $C_0[0, L]$ can be described as the subspace $V_0[0, L]$ of the space $V[0, L]$ of all functions that are right-continuous everywhere on the segment $[0, L]$. These observations, combined with the well-known result related to dual spaces equipped with the weak*-topology (see e.g. [34, II, Section 3, Proposition 7]), allow us to determine the dual to the space $V_0[0, L]$.

Proposition 4.1. *The dual space to $V_0[0, L]$, $L > 0$, which is equipped with the weak * topology generated by $C_0[0, L]$, coincides with $C_0[0, L]$.*

More precisely, if $f \in V_0[0, L]$ and $M \in C_0[0, L]$, then the duality is given by

$$\langle f, M\rangle = \int_0^L M\, df.$$

Recall that V is the vector space of right-continuous functions $f : [0, \infty) \to \mathbb{R}$ of bounded variation (see Section 2.4). Then, for any $L > 0$, the space

$$V_L := \{f \in V : f(t) = 0 \text{ for } t > L\}$$

can be identified in a natural way with $V_0[0, L]$. Note that if f is a measurable function on $[0, \infty)$ then its distribution function $d_x \in V_L$ iff $\|x\|_\infty \le L$.

For $1 \le p < q < \infty$ and $L > 0$, we define the set

$$D_{p,q,L} = \left\{\in V_L : \int_0^L Q\, df \ge 0 \text{ for all } Q \in C_0[0, L]\right.$$
$$\left.\text{which are increasing, } p\text{-convex and } q\text{-concave}\right\}.$$

Note that if $\{f_n\} \subseteq D_{p,q,L}$ is such that $f_n \xrightarrow{w^*} f \in V_L$, then by Proposition 4.1, $\int_0^L Q\, df_n \to \int_0^L Q\, df$ for all $Q \in C_0[0, L]$. Hence, it follows that $D_{p,q,L}$ is weak*-closed in V_L. Also note that $D_{p,q,L}$ is a cone, i.e., if $f \in D_{p,q,L}$, then $\alpha f \in D_{p,q,L}$ for all $\alpha \ge 0$.

Next, we will need two lemmas proved in [32] (see there Lemmas 25 and 26). However, for completeness, we present here the proof of the second of them.

Lemma 4.2. *Suppose E is a symmetric space, $E \ne L_\infty$. Then for every $\epsilon > 0$, there exists a strictly increasing continuous function $N : [0, \infty) \to [0, \infty)$, $N(0) = 0$, such that if $\int_0^1 N(|x(t)|)\, dt \le 1$, then $\|x\|_E \le \epsilon$.*

As above (see Section 2.4), for every symmetric space E we set:

$$B_c^{\le} = \{d_x : \|x\|_E \le c\} \quad \text{and} \quad B_c^{\ge} = \{d_x : \|x\|_E \ge c\}.$$

Lemma 4.3. *Suppose E is a symmetric space with $E \ne L_\infty$ and that $L > 0$. Then, for every $c > 0$, the sets $B_c^{\ge} \cap V_L$ and $B_c^{\le} \cap V_L$ are weak *-compact in V_L.*

Proof. Since the closed unit ball of the space V_L is weakly* compact and metrizable in the weak* topology (see e.g. [35, Theorems I.4.3 and I.3.16]), it is sufficient to work with an arbitrary sequence $\{x_m\}$ such that $\{d_{x_m}\} \subseteq B_c^{\geq} \cap V_L$. In other words, we have a sequence $\{d_{x_m}\}$ of the distribution functions related to given functions $x_m \geq 0$ such that $\|x_m\|_E \geq c$ and $\|x_m\|_\infty \leq L$. Note that $d_{x_m}(\tau) = 0$ if $\tau > L$. By the Helly Selection Theorem (see, for instance, [33, p. 209] or [20, VI, Section 6, Theorems 2 and 3]), we can select a subsequence $\{d_{x_{m_k}}\} \subset \{d_{x_m}\}$, which pointwise converges on the interval $[0, L]$ to some non-negative, non-increasing, right-continuous function $g(\tau)$. Set $g(\tau) = 0$ if $\tau > L$. Since $g(0) = 1$, it follows that $g(\tau)$ is the distribution function $d_x(\tau)$ of a function $x \geq 0$, $\|x\|_\infty \leq L$. By [6, Lemma 3.1], we have that $x_{m_k}^* \to x^*$ almost everywhere at all points of continuity of the function x^* on $[0, 1]$. Since $x_{m_k}^*(t) \leq L$ and $x^*(t) \leq L$ for all $t \in [0, 1]$, it follows from Lebesgue's Dominated Convergence Theorem that for any function $M \in C_0[0, L]$

$$\int_0^1 M(x_{m_k}^*(t) - x^*(t))\,dt \to 0$$

and also that

$$\begin{aligned}\int_0^L M(\tau)\,d(d_{x_{m_k}}(\tau)) &= -\int_0^1 M(x_{m_k}(t))dt \\ &\to -\int_0^1 M(x(t))dt = \int_0^L M(\tau)\,d(d_x(\tau))\end{aligned}$$

for any $M \in C_0[0, L]$. By Lemma 4.2, the first convergence yields that $\|x_{m_k}^* - x^*\|_E \to 0$, and since by the hypothesis, $\|x_{m_k}\|_E \geq c$, it follows that $\|x\|_E \geq c$. The second convergence shows that d_x is the weak* limit of the sequence $\{d_{x_{m_k}}\}$ in the space V_L. Therefore, $x \in B_c^{\geq} \cap V_L$, and we conclude that the set $B_c^{\geq} \cap V_L$ is weakly* compact. A similar proof works also for the set $B_c^{\leq} \cap V_L$. □

The following theorem is the main result of this section.

Theorem 4.4. *If E is a distributionally concave symmetric space with an order semicontinuous norm, such that $E \in Int(L_p, L_q)$, $1 \leq p < q \leq \infty$, then there exists a constant $c \in (0, 1)$ such that for every $x \in E$ with $\|x\|_E = 1$, there is an increasing, p-convex, q-concave function $M : [0, \infty) \to [0, \infty)$ such that $\int_0^1 M(|x|)\,ds \leq c^{-1}$ and $\int_0^1 M(|y|)\,ds \geq c$ whenever $\|y\|_E \geq c^{-1}$.*

Moreover, a symmetric space E is distributionally concave if and only if there exists a family of increasing, p-convex, q-concave functions $M_\alpha : [0, \infty) \to [0, \infty)$, $\alpha \in \mathcal{A}$ such that E is a subspace of the space $U(L_{M_\alpha})_{\alpha \in \mathcal{A}}$.

Proof. *The case $q < \infty$.*

We start by showing that the constant $K_1 := \lambda_0^3 > 1$, where λ_0 is the constant from Theorem 2.7, has the following property: if $L > 1$ and $0 \neq x \in E$ with $\|x\|_E = 1$, then there exists an increasing, p-convex, q-concave on $[0, L]$ function $M_L \in V_L$ such that

$$\int_0^1 M_L(|x|)\,ds \leq 1 \tag{4.1}$$

and

$$\int_0^1 M_L(|y|)\,ds \geq 1 \quad \text{if } \|y\|_\infty \leq L \text{ and } \|y\|_E \geq K_1. \tag{4.2}$$

To prove this we shall check that $\overline{\mathrm{conv}}(B^{\geq}_{K_1}) \cap V_L$ does not intersect with the algebraic sum $d_x\chi_{[0,L]} + D_{p,q,L}$, where $\overline{\mathrm{conv}}(B^{\geq}_{K_1})$ is the weakly* closed convex hull of the set $B^{\geq}_{K_1}$. Without loss of generality, we shall assume that the constant c'_E of distributional concavity of E satisfies the inequality: $c'_E \geq \lambda_0^{-1}$. From Proposition 2.3 (see also its proof) it follows that $\mathrm{conv}(B^{\geq}_{K_1}) \cap V_L \subseteq B^{\geq}_{c'_E K_1} \cap V_L$. Therefore, by Lemma 4.3, we obtain

$$\overline{\mathrm{conv}}(B^{\geq}_{K_1}) \cap V_L \subseteq B^{\geq}_{c'_E K_1} \cap V_L. \tag{4.3}$$

By the definition of the set $D_{p,q,L}$, we have $\int_0^L Q\,df \geq 0$ for all increasing, p-convex and q-concave $Q \in C_0[0, L]$ and every $f \in D_{p,q,L}$. Hence,

$$-\int_0^L Qd(f + d_x) \leq -\int_0^L Qd(d_x) = \int_0^1 Q(|x(s)|)ds. \tag{4.4}$$

Suppose that there exists an element $f \in D_{p,q,L}$ such that

$$f + d_x\chi_{[0,L]} =: d_y \in \overline{\mathrm{conv}}(B^{\geq}_{K_1}) \cap V_L.$$

Due to (4.3), it follows that $\|y\|_E \geq K_1 c'_E$ and (4.4) becomes

$$\int_0^1 Q(|y(s)|)\,ds = -\int_0^L Q\,d(d_y) \leq \int_0^1 Q(|x(s)|)\,ds.$$

We may now apply Theorem 2.7 to get

$$\lambda_0 \leq \lambda_0^2 c'_E \leq \lambda_0^{-1}\,\|y\|_E \leq \|x\|_E = 1,$$

which contradicts the assumption $\lambda_0 > 1$. Thus,

$$\left(d_x\chi_{[0,L]} + D_{p,q,L}\right) \cap \left(\overline{\mathrm{conv}}(B^{\geq}_{K_1}) \cap V_L\right) = \emptyset. \tag{4.5}$$

From Lemma 4.3 and embedding (4.3) it follows that the set $\overline{\mathrm{conv}}(B^{\geq}_{K_1}) \cap V_L$ is weakly* compact in V_L. Therefore, by (4.5), Proposition 4.1 and the Hahn–Banach separation theorem for topological vector spaces, we see that there exists $M_L \in C_0[0, L]$ such that for some constant S, we have

$$\int_0^1 M_L(|y|)\,ds = -\int_0^L M_L\,d(d_y) \geq S \quad \text{for } d_y \in \overline{\mathrm{conv}}(B^{\geq}_{K_1}) \cap V_L \tag{4.6}$$

and

$$-\int_0^L M_L\,d(d_x + f) \leq S \quad \text{for } f \in D_{p,q,L}. \tag{4.7}$$

Substituting $f = 0$ in (4.7), we have

$$\int_0^1 M_L(|x|)\,ds = -\int_0^L M_L\,d(d_x) \leq S. \tag{4.8}$$

Moreover, since $D_{p,q,L}$ is a cone, we obtain

$$-\int_0^L M_L\,d(d_x) - \alpha\int_0^L M_L\,df \leq S \quad (\text{for all } \alpha \in \mathbb{R}^+).$$

Combining this with (4.8), we see that

$$\int_0^L M_L\,df \geq 0 \quad \text{for all } f \in D_{p,q,L}. \tag{4.9}$$

Now, for any $0 \leq a < b \leq L$, consider the functions

$$g_{a,b,1}(s) = \begin{cases} -1 & \text{if } a \leq s < b \\ 0 & \text{otherwise} \end{cases}$$

$$g_{a,b,2}(s) = \begin{cases} 1 & \text{if } a^{1/p} \leq s < \left(\frac{a+b}{2}\right)^{1/p} \\ -1 & \text{if } \left(\frac{a+b}{2}\right)^{1/p} \leq s < b^{1/p} \\ 0 & \text{otherwise} \end{cases}$$

$$g_{a,b,3}(s) = \begin{cases} -1 & \text{if } a^{1/q} \leq s < \left(\frac{a+b}{2}\right)^{1/q} \\ 1 & \text{if } \left(\frac{a+b}{2}\right)^{1/q} \leq s < b^{1/q} \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that $g_{a,b,1}, g_{a,b,2}, g_{a,b,3} \in D_{p,q,L}$. It is easy to see that all three functions are in V_L (recall that $L > 1$). Let $Q \in C_0[0, L]$ be an increasing, p-convex and q-concave function. For $g_{a,b,1}$, we have

$$\int_0^L Q\,dg_{a,b,1} = Q(b) - Q(a) \geq 0 \quad \text{(since } Q \text{ is increasing).}$$

Analogously,

$$\int_0^L Q\,dg_{a,b,2} = Q(a^{1/p}) - 2Q\left(\left(\frac{a+b}{2}\right)^{1/p}\right) + Q(b^{1/p}).$$

But Q is p-convex, so $Q\left(\left(\frac{a+b}{2}\right)^{1/p}\right) \leq \frac{1}{2}Q(a^{1/p}) + \frac{1}{2}Q(b^{1/p})$. Hence $\int_0^L Q\,dg_{a,b,2} \geq 0$. Similarly, from the fact that Q is q-concave, we see that $\int_0^L Q\,dg_{a,b,3} \geq 0$. Thus, the functions $g_{a,b,1}, g_{a,b,2}, g_{a,b,3} \in D_{p,q,L}$.

From (4.9), it follows that $\int_0^L M_L\,dg_{a,b,i} \geq 0$ for every $i = 1, 2, 3$. Therefore, since a and b may be chosen arbitrarily such that $0 \leq a < b \leq L$, using arguments quite similar to those from the previous paragraph, we get that M_L is an increasing, p-convex and q-concave function on $[0, L]$. Moreover, since $M_L(0) = 0$, fixing $a = 0$ in the inequality $\int_0^L M_L\,dg_{a,b,1} \geq 0$ shows that M_L is also non-negative. Therefore, $\int_0^1 M_L(|x|)\,ds > 0$, as $x \neq 0$. Combining this with (4.8), we see that $S > 0$. Without loss of generality, we may assume that $S = 1$. Thus, from (4.8) and (4.6) it follows (4.1) and (4.2).

We would like to remove the constraint $\|y\|_\infty \leq L$ by finding the weak* limit of some sequence of functions related to the functions M_L. The first step in achieving this objective is to modify the functions M_L to form new functions which still possess properties (4.1) and (4.2) and, in addition, have an increasing, p-convex and q-concave majorant.

We set

$$M'_L(t) := \min(M_L(t), t^q) \quad (0 \leq t \leq L).$$

It is easy to check that, for every $L > 1$, M'_L satisfies the condition of Lemma 2.6 with constant $C = 1$. Therefore, there is an increasing, p-convex and q-concave function $U_L(t)(0 \le t \le L)$ such that with some constant $\alpha > 0$, which depends only on p and q, we have that $\alpha^{-1}M'_L \le U_L \le \alpha M'_L$. Moreover, by hypothesis, $L_q[0,1] \subset E$, whence $\|y\|_q \ge c_1\|y\|_E (y \in L_q)$ for some constant $0 < c_1 \le 1$.

Let $K > 2c_1^{-1}K_1$. Assume that $\|y\|_\infty \le L$ and $\|y\|_E \ge K$. Set $A := \{t \in [0,L] : M'_L(t) \le t^q\}$, $y_1 := y\chi_{y^{-1}(A)}$ and $y_2 := y - y_1$. We have that either $\|y_1\|_E \ge c_1^{-1}K_1 \ge K_1$ or $\|y_2\|_E \ge c_1^{-1}K_1$. In the first case, by (4.2), we obtain

$$\int_0^1 U_L(|y|)\,ds \ge \alpha^{-1}\int_0^1 M_L(|y_1|)\,ds \ge \alpha^{-1}.$$

In the second case, supposing $\|y_2\|_E \ge c_1^{-1}K_1$, we still have

$$\int_0^1 U_L(|y|)\,ds \ge \alpha^{-1}\int_0^1 |y_2(s)|^q\,ds \ge \alpha^{-1}c_1^q\|y_2\|_E^q \ge \alpha^{-1}K_1^q \ge \alpha^{-1}.$$

Thus, we always have that

$$\int_0^1 U_L(|y|)\,ds \ge \alpha^{-1} \quad \text{if } \|y\|_\infty \le L \text{ and } \|y\|_E \ge K. \tag{4.10}$$

Moreover, from (4.1) it follows that

$$\int_0^1 U_L(|x|)\,ds \le \alpha \quad (L > 1). \tag{4.11}$$

Next, let us note that for every $L > 1$ the following inequality holds:

$$U_L(t) \le \alpha t^q \quad (0 \le t \le L).$$

Therefore, setting $U_L(s) := 0$ for $s \ge L$, we have that $\{\alpha^{-1}U_L(s)s^{-q}\}_{L\in\mathbb{N}}$ belongs to the unit ball of $L_\infty[0,\infty)$. Since the unit ball of $L_\infty[0,\infty)$ with the weak* topology is compact, it has a subsequence (we will denote it in the same way) which weakly* converges to some function, say, $\alpha^{-1}M(s)s^{-q}$ from this ball.

Now, we show that

$$\int_0^1 U_L(|y|)\,dt \to \int_0^1 M(|y|)\,dt \quad \text{as } L \to \infty \tag{4.12}$$

for every $y \in L_\infty$.

If $y \in L_\infty$ is arbitrary, a slight approximation allows us to assume that its distribution function d_y is absolutely continuous on the semi-axis $[0,\infty)$. Therefore,

$$-\int_0^\infty s^q \frac{d}{ds}(d_y(s))\,ds = -\int_0^\infty s^q\,d(d_y(s)) = \int_0^1 |y(t)|^q\,dt < \infty,$$

whence the function $s^q \frac{d}{ds}(d_y(s)) \in L_1[0,\infty)$. Since $M(s)s^{-q}$ is a weak* limit of $\{U_L(s)s^{-q}\}_{L\in\mathbb{N}}$ in $L_\infty[0,\infty)$, we have that

$$\int_0^\infty U_L(s)s^{-q}\cdot g(s)\,ds \to \int_0^\infty M(s)s^{-q}\cdot g(s)\,ds \quad \text{as } L \to \infty$$

for all $g \in L_1[0,\infty)$. Putting here $g(s) = s^q \frac{d}{ds}(d_y(s))$, we obtain that

$$\int_0^1 U_L(|y|)\,dt = -\int_0^\infty U_L\,d(d_y(s)) \to -\int_0^\infty M\,d(d_y(s)) = \int_0^1 M(|y|)\,dt$$

as $L \to \infty$. Therefore, (4.12) is proved.

Let $\|y\|_E \ge K$ and $y \in L_\infty$. It follows from (4.10) that $\int_0^1 U_L(|y|)\,ds \ge \alpha^{-1}$ whenever $\|y\|_E \le L$. Combining this with (4.12), we obtain that $\int_0^1 M(|y|)\,ds \ge \alpha^{-1}$ as well. Since the norm of E is order semicontinuous, a straightforward application of Lebesgue's Monotone Convergence Theorem allows us to remove the condition $y \in L_\infty$.

Furthermore, letting in (4.12) $y := s\chi_{[0,1]}$, where $s \ge 0$, we obtain that $U_L(t)$ converges to $M(t)$ pointwise on $[0,\infty)$. Therefore, M is an increasing, p-convex and q-concave function. Moreover, by (4.11), we see that $\int_0^1 M(|x|)\,ds \le \alpha$. Setting $c := \min(K^{-1}, \alpha^{-1})$, we obtain the first assertion of the theorem.

To prove the second assertion of the theorem, consider the set $\mathcal{A} := \{x \in E : \|x\|_E = 1\}$ and the family $\{M_x\}_{x\in A}$ consisting of the functions $M = M_x$, which were constructed in the first part.

Let $y \in E$, $\|y\|_E = 1$. Due to the first assertion of the theorem, there exists an increasing, p-convex and q-concave function M_y such that $\int_0^1 M_y(|y|)\,ds \le c^{-1}$. Since M_y is convex and $c < 1$, we obtain that $\int_0^1 M_y(c\,|y|)\,ds \le 1$, whence $\|y\|_{L_{M_y}} \le c^{-1}$. Therefore,

$$\|y\|_{U(L_{M_x})_{x\in\mathcal{A}}} := \inf_{x\in\mathcal{A}} \|y\|_{L_{M_x}} \le \|y\|_{L_{M_y}} \le c^{-1}.$$

Now, let us prove the converse inequality. Since $\|c^{-1}y\|_E = c^{-1}$, by the first assertion of the theorem, we have that $\int_0^1 M_x(c^{-1}\,|y|)\,ds \ge c$ for all $x \in \mathcal{A}$. Since $c < 1$, the convexity of the functions M_x implies that, for all $x \in \mathcal{A}$, $\int_0^1 M_x(c^{-2}\,|y|)\,ds \ge 1$, which implies: $\|y\|_{L_{M_x}} \ge c^2$. Therefore,

$$\|y\|_{U(L_{M_x})_{x\in\mathcal{A}}} \ge c^2,$$

and the equivalence of the norms $\|y\|_E$ and $\|y\|_{U(L_{M_x})_{x\in\mathcal{A}}}$ on E is proved.

Since every Orlicz space is distributionally concave with constant 1 (see Section 2.4), the proof of the converse assertion is straightforward. Thus, the theorem is fully proved in the case when $q < \infty$.

The case $q = \infty$.

Arguing similarly to the case of a finite q, we can prove that for every $x \in E$ such that $\|x\|_E = 1$ and for any $L > 1$ there exists an increasing and p-convex function M_L, $M_L(0) = 0$, such that relations (4.1) and (4.2) hold.

Since the case $E = L_\infty$ is trivial, we can assume that $E \ne L_\infty$. Then, by Lemma 4.2, there exists a strictly increasing continuous function $N_1 : [0,\infty) \to [0,\infty)$ with the following property: if $\int_0^1 N_1(|y(t)|)\,dt \le 1$, then $\|y\|_E \le 1$. It is easy to see that the function

$$N(t) := \sup_{0<s\le 1} \frac{N_1(st)}{s^p}$$

still has the same property as N_1. Furthermore, N satisfies the condition of Lemma 2.6 with constant $C = 1$ as

$$N(ut) = u^p \sup_{0<s\le 1} \frac{N_1(ust)}{s^p u^p} \le u^p \sup_{0<v\le 1} \frac{N_1(vt)}{v^p} = u^p N(t)$$

and so it is equivalent to a p-convex function. Therefore, without loss of generality, we may assume that the function N is convex.

Our next arguments are similar to the ones used in the case of finite q except instead of the function t^q we use the function N.

Consider the functions

$$M'_L(t) := \min(M_L(t), N(t)) \quad (0 \le t \le L).$$

For any $L > 1$, the function M'_L satisfies the inequality $M'_L(st) \le s^p M'_L(t)$ if $0 < s \le 1$. So, by Lemma 2.6, there is an increasing and p-convex function $U_L(t)(0 \le t \le L)$ such that with some constant $\beta > 0$ depending only on p, we have that $\beta^{-1} M'_L(t) \le U_L(t) \le \beta M'_L(t)(0 \le t \le L)$.

Let $K > 2K_1$. Assume that $\|y\|_\infty \le L$ and $\|y\|_E \ge K$. Set $A := \{t \in [0, L] : M'_L(t) \le N(t)\}$, $y_1 := y\chi_{y^{-1}(A)}$ and $y_2 := y - y_1$. Then we have that either $\|y_1\|_E \ge K_1$ or $\|y_2\|_E \ge K_1$. In the first case, by (4.2), we obtain

$$\int_0^1 U_L(|y|)\,ds \ge \beta^{-1} \int_0^1 M_L(|y_1|)\,ds \ge \beta^{-1}.$$

Now, suppose $\|y_2\|_E \ge K_1$. Since $\int_0^1 N(|y|)\,dt \le 1$ implies $\|y\|_E \le 1$ and N is a convex function, we have that $\|y\|_E \le \|y\|_{L_N}(y \in L_N)$. By assumption, $a := \|y_2\|_E \ge K_1 > 1$. Therefore, $\int_0^1 N(|y_2|/a)\,dt \ge 1$. Since $a > 1$, we have that $N(|y_2|/a) \le N(|y_2|)/a$, and from the last inequality it follows $\int_0^1 N(|y_2|)\,dt \ge a = \|y_2\|_E$. Hence,

$$\int_0^1 U_L(|y|)\,ds \ge \beta^{-1} \int_0^1 N(|y_2|)\,ds \ge \beta^{-1}\|y_2\|_E \ge \beta^{-1}K_1 \ge \beta^{-1}.$$

Thus, we have that

$$\int_0^1 U_L(|y|)\,ds \ge \beta^{-1} \quad \text{if } \|y\|_\infty \le L \text{ and } \|y\|_E \ge K. \tag{4.13}$$

Moreover, from (4.1) and from the inequality $U_L(t) \le \beta M_L(t)(0 \le t \le L)$ it follows that

$$\int_0^1 U_L(|x|)\,ds \le \beta. \tag{4.14}$$

Since for every $L > 0$ we have $U_L(t) \le \beta N(t)(0 \le t \le L)$, then setting $U_L(s) := 0$ for $s \ge L$, we have that some subsequence from the sequence $\left\{\frac{U_L}{\beta N}\right\}_{L\in\mathbb{N}}$ has a weak* limit in $L_\infty[0,\infty)$, say, $\frac{M}{\beta N}$ belonging to the unit ball of $L_\infty[0,\infty)$. As above, it is not hard to check that the function M satisfies all the requirements of the theorem.

The second assertion can be proved now in the same way as in the case of a finite q. □

The analogous statement holds also for distributionally convex spaces.

Theorem 4.5 ([32, Theorem 21]). *If E is a distributionally convex symmetric space with an order semicontinuous norm, such that $E \in Int(L_p, L_q)$, $1 \le p < q \le \infty$, then there exists a constant $c > 1$ such that for every $x \in E$ with $\|x\|_E = 1$, there is an increasing, p-convex and q-concave function $M : [0,\infty) \to [0,\infty)$, $M(0) = 0$, such that $\int_0^1 M(|x|)\,ds \ge c^{-1}$ and $\int_0^1 M(|y|)\,ds \le c$ whenever $\|y\|_E \le c^{-1}$.*

Moreover, a symmetric space E is distributionally convex if and only if there exists a family of increasing, p-convex, q-concave functions $M_\alpha : [0,\infty) \to [0,\infty)$, $\alpha \in A$, such that E is a subspace of the space $\Delta(L_{M_\alpha})_{\alpha\in\mathcal{A}}$.

Theorem 4.6. *If E is a distributionally concave space with an order semicontinuous norm, such that $E \in Int(L_p, L_q)$, $1 \le p < q \le \infty$, then E is q-concave and there is a constant $C > 0$ such that the inequality*

$$\left(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|_E^p\right)^{1/p} \le C\,\|C(x_1,x_2,\dots,x_n)\|_E \tag{4.15}$$

holds for any $x_1,x_2,\dots,x_n \in E$ and $n \ge 1$.

Proof. We begin by proving the inequality (4.15). By Theorem 4.4, we should prove only that there exists a constant $C_1 > 0$ such that for any increasing, p-convex, q-concave function $M : [0,\infty) \to [0,\infty)$ and for given $x_1,x_2,\dots,x_n$ from $E \cap L_M$ we have that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|_{L_M}^p\right)^{1/p} \le C_1\,\|C(x_1,x_2,\dots,x_n)\|_{L_M}. \tag{4.16}$$

The case $p = 1$.

Since the norm in an Orlicz space is order semicontinuous, we have

$$\|x\|_{L_M} = \sup\left\{\int_0^1 xy\,ds \;:\; \|y\|_{(L_M)'} \le 1\right\}.$$

It is well known (see, for instance, [21, Section 2.14]) that $(L_M)' = L_{M'}$, where M' is the complementary Orlicz function for M, and with a universal constant $\gamma > 0$ we have $\gamma^{-1}\|y\|_{L_{M'}} \le \|y\|_{(L_M)'} \le \gamma\|y\|_{L_{M'}}$. Therefore,

$$\gamma^{-1}\,\|x\|_{L_M} \le \sup\left\{\int_0^1 xy\,ds \;:\; \|y\|_{L_{M'}} \le 1\right\} \le \gamma\,\|x\|_{L_M}.$$

Hence, for every $k = 1,\dots,n$, there exists a y_k such that $\|y_k\|_{L_{M'}} \le 1$ and $\frac{1}{2\gamma}\|x_k\|_{L_M} \le \int_0^1 x_k y_k\,ds$. Since the Orlicz space $L_{M'}$ is distributionally convex with constant 1 (see Section 2.4) it follows that

$$\|C(y_1,y_2,\dots,y_n)\|_{L_{M'}} \le \max_{k=1,\dots,n}\|y_k\|_{L_{M'}} \le 1.$$

Therefore, we have

$$\begin{aligned}\frac{1}{2\gamma}\cdot\frac{1}{n}\sum_{k=1}^{n}\|x_k\|_{L_M} &\le \frac{1}{n}\sum_{k=1}^{n}\int_0^1 x_k y_k\,ds\\ &= \int_0^1 C(x_1,x_2,\dots,x_n)\cdot C(y_1,y_2,\dots,y_n)\,ds\\ &\le \|C(x_1,x_2,\dots,x_n)\|_{L_M}\,\|C(y_1,y_2,\dots,y_n)\|_{L_{M'}}\\ &\le \|C(x_1,x_2,\dots,x_n)\|_{L_M},\end{aligned}$$

that is,

$$\frac{1}{n}\sum_{k=1}^{n}\|x_k\|_{L_M} \le C_1 \|C(x_1, x_2, \dots, x_n)\|_{L_M}, \tag{4.17}$$

where $C_1 = 2\gamma$.

The case $p > 1$.

For an increasing, p-convex, q-concave function $M : [0, \infty) \to [0, \infty)$, we set

$$M_p(t) := M(t^{1/p}).$$

Clearly, M_p is 1-convex. For any $x_k \in L_M, x_k \ge 0$, we have that $x_k^p \in L_{M_p} (k = 1, 2, \dots, n)$. Therefore, applying (4.17) to x_k^p in L_{M_p}, we obtain

$$\frac{1}{n}\sum_{k=1}^{n}\left\|x_k^p\right\|_{L_{M_p}} \le C_1 \left\|C(x_1^p, x_2^p, \dots, x_n^p)\right\|_{L_{M_p}}. \tag{4.18}$$

Noting that for an arbitrary $x \ge 0$, we have

$$\begin{aligned}\left\|x^p\right\|_{L_{M_p}}^{1/p} &= \inf\left\{\rho > 0 : \int_0^1 M_p\left(\frac{x^p}{\rho}\right) dt \le 1\right\}^{1/p} \\ &= \inf\left\{\rho^{1/p} > 0 : \int_0^1 M\left(\frac{x}{\rho^{1/p}}\right) dt \le 1\right\} \\ &= \|x\|_{L_M},\end{aligned}$$

we observe that (4.18) is actually equivalent to

$$\frac{1}{n}\sum_{k=1}^{n}\|x_k\|_{L_M}^p \le C_1 \|C(x_1, x_2, \dots, x_n)\|_{L_M}^p.$$

This proves (4.16) (and (4.15)).

To prove the q-concavity of E we need to show that there is a constant $C' > 0$ such that for any $x_1, x_2, \dots, x_n$ from E the inequality

$$\left(\sum_{k=1}^{n}\|x_k\|_E^q\right)^{1/q} \le C' \left\|\left(\sum_{k=1}^{n}|x_k|^q\right)^{1/q}\right\|_E$$

holds. Once again, it suffices to prove that for any increasing, p-convex, q-concave function $M : [0, \infty) \to [0, \infty)$ and for given $x_1, x_2, \dots, x_n$ from $E \cap L_M$ we have that

$$\left(\sum_{k=1}^{n}\|x_k\|_{L_M}^q\right)^{1/q} \le \left\|\left(\sum_{k=1}^{n}|x_k|^q\right)^{1/q}\right\|_{L_M}.$$

However, the latter inequality is an immediate consequence of the q-concavity of the function M. □

From the last theorem, we can deduce that the concept of distributionally concavity from [32] coincides with that from [38] at least for symmetric spaces E such that $E \in Int(L_1, L_\infty)$ (in particular, if E is separable or E has the Fatou property).

Corollary 4.7. *A symmetric space E with $E \in Int(L_1, L_\infty)$ is distributionally concave if and only if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and for given $x_1, x_2, \ldots, x_n$ from E we have that*

$$\|C(x_1, \ldots, x_n)\|_E \geq \frac{c}{n} \sum_{k=1}^{n} \|x_k\|_E .$$

In particular, from Corollary 4.7 and [38] it follows that the concepts of distributional convexity and distributional concavity are (Köthe) dual to each other.

In the case of symmetric spaces with the Fatou property we are able to obtain a more precise result than Theorem 4.4.

Corollary 4.8. *If E is a distributionally concave symmetric space with the Fatou property such that $E \in Int(L_p, L_q)$, $1 \leq p < q \leq \infty$, then there exists a family of increasing, p-convex, q-concave functions $M_\alpha : [0, \infty) \to [0, \infty)$, $\alpha \in \mathcal{A}$ such that $E = U(L_{M_\alpha})_{\alpha \in \mathcal{A}}$ (with equivalence of norms).*

Proof. By Theorem 4.4, we have that

$$E \subseteq U(L_{M_\alpha})_{\alpha \in \mathcal{A}} \tag{4.19}$$

and for every $x \in E$

$$c^2 \|x\|_E \leq \|x\|_{U(L_{M_\alpha})_{\alpha \in \mathcal{A}}} := \inf_{\alpha \in \mathcal{A}} \|x\|_{L_{M_\alpha}} \leq c^{-1} \|x\|_E, \tag{4.20}$$

where $c \in (0, 1)$ is a constant from Theorem 4.4. Let us show that

$$U(L_{M_\alpha})_{\alpha \in \mathcal{A}} \subseteq E. \tag{4.21}$$

Assuming that $x \in U(L_{M_\alpha})_{\alpha \in \mathcal{A}}$ and $x \geq 0$, we set: $x_m := \min(x, m)$, $m \geq 1$. Clearly, $\|x_m\|_{U(L_{M_\alpha})_{\alpha \in \mathcal{A}}} \leq \|x\|_{U(L_{M_\alpha})_{\alpha \in \mathcal{A}}}$ for all $m \geq 1$. Therefore, by (4.20), $\sup_{m \in \mathbb{N}} \|x_m\|_E \leq c^{-2} \|x\|_{U(L_{M_\alpha})_{\alpha \in \mathcal{A}}} < \infty$. Since $x_m \uparrow x$ and E has the Fatou property, we obtain that $x \in E$ and $\|x\|_E \leq c^{-2} \|x\|_{U(L_{M_\alpha})_{\alpha \in \mathcal{A}}}$. Relations (4.19) and (4.21) show that the proof is complete. □

Let us prove an analogous statement for distributionally convex symmetric spaces.

Corollary 4.9. *If E is a distributionally convex symmetric space with the Fatou property such that $E \in Int(L_p, L_q)$, $1 \leq p < q \leq \infty$, then there exists a family of increasing, p-convex, q-concave functions $N_\alpha : [0, \infty) \to [0, \infty)$, $\alpha \in \mathcal{A}$, such that $E = \Delta_{\alpha \in \mathcal{A}} L_{N_\alpha}$.*

Proof. Since E' is distributionally concave and since $E' \in Int(L_{q'}, L_{p'})$ (here, of course, $1/q + 1/q' = 1$ and $1/p + 1/p' = 1$), we conclude by appealing to the preceding corollary that $E' = U(L_{M_\alpha})_{\alpha \in \mathcal{A}}$, where $\{M_\alpha\}_{\alpha \in \mathcal{A}}$ is a family of increasing q'-convex and p'-concave functions. Therefore, $U(L_{M_\alpha})_{\alpha \in \mathcal{A}}$ is a Banach space with the norm

$$\|x\|_{U(L_{M_\alpha})_{\alpha \in \mathcal{A}}} := \inf_{\alpha \in \mathcal{A}} \|x\|_{L_{M_\alpha}} .$$

Hence, by Theorem 2.11, we have

$$E' = \sum_{\alpha \in \mathcal{A}} L_{M_\alpha}.$$

By duality, we infer immediately that

$$E'' = \Delta_{\alpha\in\mathcal{A}}(L_{M_\alpha})' = \Delta_{\alpha\in\mathcal{A}} L_{M'_\alpha},$$

where M'_α is the complementary Orlicz function to M_α. Since E has the Fatou property, we have $E = E''$. Finally, since M_α is q'-convex and p'-concave, we see that $N_\alpha = M'_\alpha$ is a p-convex and q-concave function. □

Finally, we shall combine the preceding results.

Corollary 4.10. *If E is a distributionally convex and distributionally concave symmetric space with an order semicontinuous norm such that $E \in Int(L_1, L_\infty)$, then there exists an increasing, convex function $M : [0,\infty) \to [0,\infty)$ such that E is a subspace of the Orlicz space L_M. In particular, if E has the Fatou property, then $E = L_M$.*

Proof. By Theorems 4.4 and 4.5, there are families of increasing convex functions $M_\alpha : [0,\infty) \to [0,\infty)$, $\alpha \in \mathcal{A}$ and $N_\beta : [0,\infty) \to [0,\infty)$, $\beta \in \mathcal{B}$ such that

$$\|x\|_E \asymp \sup_{\alpha\in\mathcal{A}} \|x\|_{L_{M_\alpha}} \asymp \inf_{\beta\in\mathcal{B}} \|x\|_{L_{N_\beta}}, \quad \forall x \in E. \tag{4.22}$$

Let $C_1 \geq 1$ be the constant of both equivalences from above. Thus, in particular, for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $x \in E$, we have

$$\|x\|_{M_\alpha} \leq C_1 \|x\|_{N_\beta}.$$

From this observation and from [21, Section 2.13] we infer that

$$M_\alpha(s) \leq N_\beta(C_2 s), \quad \forall \alpha \in \mathcal{A},\ \beta \in \mathcal{B},\ s > 0 \tag{4.23}$$

with some constant $C_2 > 0$ depending only on C_1.

Denote

$$M(s) := \sup_{\alpha\in\mathcal{A}} M_\alpha(s), \quad s > 0.$$

Clearly, M is an increasing convex function on $[0,\infty)$, and from (4.23) it follows

$$M(s) \leq N_\beta(C_2 s), \quad \forall \beta \in \mathcal{B},\ s \geq 0.$$

Therefore, by (4.22), for every $x \in E$, we have

$$\|x\|_E \leq C_1 \sup_{\alpha\in\mathcal{A}} \|x\|_{L_{M_\alpha}} \leq C_1 \|x\|_{L_M} \leq C_1 C_2 \inf_{\beta\in\mathcal{B}} \|x\|_{L_{N_\beta}} \leq C_1^2 C_2 \|x\|_E.$$

The last assertion follows now from Corollaries 4.8 and 4.9. □

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References

[1] T. Andô, On products of Orlicz spaces, Math. Ann. 140 (1960) 174–186.
[2] S.V. Astashkin, On a bilinear multiplicative operator, in: Issled. Teor. Funkst. Mnogikh Veshchestv. Perem., Yaroslavl' Gos. Univ., Yaroslavl', 1982, pp. 3–15 (in Russian).

[3] S.V. Astashkin, Interpolation of positive polylinear operators in Calderón–Lozanovskiĭ spaces, Sibirsk. Mat. Zh. 38 (6) (1997) 1211–1218 (in Russian); English translation in Siberian Math. J. 38 (6) (1997), 1047–1053.

[4] S.V. Astashkin, Tensor product in symmetric function spaces, in: Proc. of Fourth International Conference on Function Spaces (Zielona Gora, 1995), Collect. Math. 48 (4–6) (1997) 375–391.

[5] S.V. Astashkin, On multiplier of a rearrangement invariant space with respect to the tensor product, Funktsional. Anal. i Prilozhen 30 (4) (1996) 58–60 (in Russian); English translation in Funct. Anal. Appl. 30 (4) (1996), 267–269.

[6] S.V. Astashkin, Rademacher functions in symmetric spaces, Sovrem. Mat. Fundam. Napravl. 32 (2009) 3–161 (in Russian); English translation in J. Math. Sci. (New York) 169 (6) (2010) 725–886.

[7] S.V. Astashkin, L. Maligranda, Interpolation between L_1 and L_p, $1 < p < \infty$,, Proc. Amer. Math. Soc. 132 (10) (2004) 2929–2938.

[8] S.V. Astashkin, F.A. Sukochev, Banach–Saks property in Marcinkiewicz spaces, J. Math. Anal. Appl. 336 (2) (2007) 1231–1258.

[9] S.V. Astashkin, F.A. Sukochev, C.P. Wong, Disjointification of martingale differences and conditionally independent random variables with some applications, Studia Math. 205 (2) (2011) 171–200.

[10] C. Bennett, R. Sharpley, Interpolation of operators, in: Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988.

[11] N. Bourbaki, Topological vector spaces, in: Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987, Translated from the French by H. G. Eggleston and S. Madan.

[12] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. 24 (1964) 113–190.

[13] N.L. Carothers, Rearrangement invariant subspaces of Lorentz function spaces, Israel J. Math. 40 (3–4) (1981) 217–228.

[14] N.L. Carothers, Rearrangement invariant subspaces of Lorentz function spaces II, Rocky Mountain J. Math. 17 (3) (1987) 607–616.

[15] W.B. Johnson, B. Maurey, G. Schechtman, L. Tzafriri, Symmetric structures in Banach spaces, Mem. Amer. Math. Soc. 19 (217) (1979).

[16] N. Kalton, Representations of operators between function spaces, Indiana Univ. Math. J. 33 (5) (1984) 639–665.

[17] N. Kalton, S.J. Montgomery-Smith, Interpolation of Banach spaces, in: Handbook of the Geometry of Banach Spaces, vol. 2, North-Holland, Amsterdam, 2003, pp. 1131–1175.

[18] A. Kaminska, Some remarks on Orlicz–Lorentz spaces, Math. Nachr. 147 (1) (1990) 29–38.

[19] A. Kaminska, Y. Raynaud, Isomorphic copies in the lattice E and its symmetrization $E^{(*)}$ with applications to Orlicz–Lorentz spaces, J. Funct. Anal. 257 (1) (2009) 271–331.

[20] A.N. Kolmogorov, S.V. Fomin, Elements of the Theory of Functions and Functional Analysis, 7th ed., Fizmatlit, Moscow, 2004 (in Russian).

[21] M.A. Krasnosel'skii, Ja.B. Rutickii, Vypuklye funkcii i prostranstva Orlicza, Fizmatgiz, Moscow, 1958 (in Russian); English translation, Convex functions and Orlicz spaces, Noordhoff, Groningen, 1961.

[22] S.G. Krein, Ju.I. Petunin, E.M. Semenov, Interpolation of linear operators, in: Translations of Mathematical Monographs, vol. 54, Amer. Math. Soc., 1982.

[23] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces. I. Sequence spaces, in: Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 92, Springer-Verlag, Berlin-New York, 1977.

[24] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces. II. Function spaces, in: Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97, Springer-Verlag, Berlin-New York, 1979.

[25] G.G. Lorentz, T. Shimogaki, Interpolation theorems for the pairs of spaces (L^p, L^∞) and (L^1, L^q), Trans. Amer. Math. Soc. 159 (1971) 207–221.

[26] G. Ja Lozanovskiĭ, On some Banach lattices IV, Sibirsk. Mat. Z. 14 (1973) 140–155 (in Russian); English translation in Siberian Math. J. 14 (1973), 97–108.

[27] G.Ja. Lozanovskiĭ, Transformations of ideal Banach spaces by means of concave functions, in: Qualitative and Approximate Methods for the Investigation of Operator Equations, Yaroslavl, 1978, pp. 122–147 (in Russian).

[28] F. Lust-Piquard, F. Sukochev, The p-Banach Saks property in symmetric operator spaces, Illinois J. Math. 51 (4) (2007) 1207–1229.

[29] L. Maligranda, Positive bilinear operators in Calderón–Lozanovskiĭ spaces, Arch. Math. 81 (2003) 26–37.

[30] J. Martin, M. Milman, Extrapolation methods and Rubio de Francia's extrapolation theorem, Adv. Math. 201 (2006) 209–262.

[31] M. Milman, Embeddings of Lorentz–Marcinkiewicz spaces with mixed norms, Anal. Math. 4 (1978) 215–223.

[32] S. Montgomery-Smith, E. Semenov, Random rearrangements and operators, in: Voronezh Winter Mathematical Schools, in: Amer. Math. Soc. Transl. Ser. 2, vol. 184, Amer. Math. Soc., Providence, RI, 1998, pp. 157–183.

[33] I.P. Natanson, Theory of Functions of Real Variable M, Nauka, 1974 (in Russian).

[34] A.P. Robertson, W. Robertson, Topological vector spaces, in: Cambridge Tracts in Mathematics, vol. 53, Cambridge University Press, Cambridge-New York, 1980 (Reprint of the second edition).

[35] W. Rudin, Functional analysis, in: McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, Düsseldorf, Johannesburg, 1973.

[36] E.M. Semenov, F.A. Sukochev, The Banach–Saks index, Mat. Sb. 195 (2) (2004) 117–140 (in Russian); English translation in Sb. Math. 195 (1–2) (2004), 263–285.

[37] V. Shestakov, Transformations of Banach ideal spaces and interpolation of linear operators, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (11–12) (1981) 569–577 (in Russian).

[38] F.A. Sukochev, RUC-bases in Orlicz and Lorentz operator spaces, Positivity 2 (3) (1998) 265–279.