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BANACH-SAKS TYPE PROPERTIES IN REARRANGEMENT-INVARIANT SPACES WITH THE KRUGLOV PROPERTY

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Abstract. We study Banach-Saks index sets for rearrangement-invariant (r. i.) function spaces. The main focus is on the class of r. i. spaces whose (classical) Banach-Saks index set is trivial, but for which the counterpart computed for weakly null sequences of independent random variables is not trivial. We discover and exploit an interesting connection with the Kruglov property in r. i. spaces.

Introduction

Let X be a Banach space and $p \geq 1$. A bounded sequence $\{x_n\} \subset X$ is called a p-BS-sequence (BS-sequence) if there exists a subsequence $\{y_k\} \subset \{x_n\}$ such that

$$C = \sup_{m \in N} m^{-\frac{1}{p}} \Big\| \sum_{k=1}^{m} y_k \Big\|_X < \infty \quad \left(\lim_{m \to \infty} \frac{1}{m} \Big\| \sum_{k=1}^{m} y_k \Big\|_X = 0 \right).$$

We say that X has the p-BS-property (BS-property) and we write $X \in BS(p)$ $(X \in (BS))$ if each weakly null sequence contains a p-BS-sequence (BS-sequence). Clearly, every Banach space has the 1-BS-property. The set

$$\Gamma(X) = \{p : \; p \geq 1, \; X \in BS(p)\}$$

is said to be the index set of X, and is of the form $[1, \gamma]$, or $[1, \gamma)$ for some $1 \leq \gamma$. The Banach-Saks index $\gamma(X)$ of X is said to be γ in the first case or $\gamma - 0$ in the

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second case. Everywhere below X stands for an rearrangement-invariant (briefly, r.i.) space on $[0,1]$. In particular, classical results [5, Ch. 12, Theorem 2], [13] say that $\Gamma(L_p) = [1, \min(p, 2)]$ for any $p \in [1, \infty)$. Every Banach r.i. space $X \in (BS)$ is automatically separable (see e.g. [20]).

If, in the preceding definition, we replace all weakly null sequences by weakly null sequences of independent random variables (respectively, by weakly null sequences of pairwise disjoint elements), we obtain the set $\Gamma_{\mathrm{i}}(X)$ (respectively, $\Gamma_{\mathrm{d}}(X)$). It is known [1] that $1 \in \Gamma(X) \subset \Gamma_i(X) \subset [1, 2]$ and $\Gamma_i(X) \subset \Gamma_d(X)$ for any r. i. space X. Moreover, the sets $\Gamma(X)$ and $\Gamma_i(X)$ coincide in many cases but not always. For example, $\Gamma(L_p) = \Gamma_i(L_p)$, $1 < p < \infty$ (see e.g. [21, Theorem 9]), whereas for the Lorentz space $L_{2,1}$ generated by the function $t^{1/2}$, we have $\Gamma(L_{2,1}) = [1, 2)$ and $\Gamma_i(L_{2,1}) = [1, 2]$ ([20, Theorem 5.9] and [1, Proposition 4.12]). It turns out that these two situations are typical: under the assumption that $\Gamma(X) \neq \{1\}$, we have either $\Gamma_i(X) \setminus \Gamma(X) = \emptyset$ or else $\Gamma_i(X) \setminus \Gamma(X) = \{2\}$.

Theorem 0.1. *(*[21, Theorem 9]*) If X is a separable r. i. space such that $\Gamma(X) \neq \{1\}$, then either*

(i) $\Gamma(X) = \Gamma_i(X)$, *or*
(ii) $\Gamma(X) = [1, 2)$ *and* $\Gamma_i(X) = [1, 2]$.

We show in the present article that the assumption $\Gamma(X) \neq \{1\}$ above is crucial and, in the sharp contrast with the result of Theorem 0.1, for any given $1 < p \leq 2$ there exists a r. i. space X such that $\Gamma_i(X) \setminus \Gamma(X) = (1, p]$. Furthermore, the proof of Theorem 9 in [21] shows that the assumption $\Gamma(X) \neq \{1\}$ in Theorem 0.1 above can be replaced by a weaker one: the lower Boyd index $\alpha_X > 0$ [1, Theorem 4.2(i)]. In this article, we shall prove that there exists a reflexive r. i. space X such that $\Gamma_i(X) = [1, 2]$ and $\alpha_X = 0$. Our approach depends crucially on an interesting link between Banach-Saks type properties in r. i. spaces and the so-called Kruglov property which has been studied and intensively used in [7, 2, 3, 4]. These problems are considered in Section 2. In Section 3 we deal with r. i. spaces having Banach-Saks index 2 showing, in particular, that L_2 is unique r. i. space with index 2 and containing $L_{2,1}$.

1. Definitions and preliminaries

1.1. **Rearrangement-invariant spaces.** A Banach space $(X, \|\cdot\|_X)$ of real-valued Lebesgue measurable functions (with identification m-a.e.) on the interval $[0, 1]$ will be called *rearrangement-invariant* (briefly, r. i.) if

(i). X is an ideal lattice, that is, if $y \in X$, and if x is any measurable function on $[0, 1]$ with $0 \leq |x| \leq |y|$ then $x \in X$ and $\|x\|_X \leq \|y\|_X$;

(ii). X is rearrangement invariant in the sense that if $y \in X$, and if x is any measurable function on $[0,1]$ with $x^* = y^*$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

Here, m denotes Lebesgue measure and x^* denotes the non-increasing, right-continuous rearrangement of x given by

$$x^*(t) = \inf\{\ s \geq 0 : m(\{u \in [0,1] : |\ x(u)\ | > s\}) \leq t\ \}, \quad t > 0.$$

For basic properties of r.i. spaces, we refer to the monographs [15, 17]. We note that for any r.i. space X we have: $L_\infty[0,1] \subseteq X \subseteq L_1[0,1]$. Without loss of generality, we shall assume everywhere below that these embeddings hold with constant 1. Recall that for $0 < \tau < \infty$, the dilation operator σ_τ is defined by setting

$$\sigma_\tau x(t) = \begin{cases} x(t/\tau),\ 0 \leq t \leq \min(1,\tau) \\ 0,\ \min(1,\tau) < t \leq 1. \end{cases}$$

The dilation operators σ_τ are bounded in every r. i. space X. The numbers α_X and β_X given by

$$\alpha_X := \lim_{\tau \to 0} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}, \quad \beta_X := \lim_{\tau \to \infty} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}$$

belong to the closed interval $[0,1]$ and are called the Boyd indices of X.

The Köthe dual $X^\times$ of an r.i. space X on $[0,1]$ consists of all measurable functions y for which

$$\|y\|_{X^\times} := \sup\left\{ \int_0^1 |x(t)y(t)|\, dt :\ x \in X,\ \|x\|_X \leq 1 \right\} < \infty.$$

If X^* denotes the Banach dual of X, then $X^\times \subset X^*$ and $X^\times = X^*$ if and only if X is separable. An r.i. space X is said to have the *Fatou property* if whenever $\{f_n\}_{n=1}^\infty \subseteq X$ and f measurable on $[0,1]$ satisfy $f_n \to f$ a.e. on $[0,1]$ and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$. It is well-known that an r.i. space X has the Fatou property if and only if the natural embedding of X into its Köthe bidual $X^{\times\times}$ is a surjective isometry.

Let us recall some classical examples of r.i. spaces on $[0,1]$. Denote by Ω the set of all increasing concave functions on $[0,1]$ with $\varphi(0) = \varphi(+0) = \lim\limits_{t\to 0} \frac{t}{\varphi(t)} = 0$. Each function $\varphi \in \Omega$ generates the Lorentz space $\Lambda(\varphi)$ (see e.g. [15]) endowed with the norm

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t) d\varphi(t).$$

The Lorentz space $\Lambda(\varphi)$, where $\varphi(t) := t^{1/2}$, $t \geq 0$ is customarily denoted $L_{2,1}$.

Let $M(t)$ be a convex function on $[0,\infty)$ such that $M(t)>0$ for all $t>0$ and such that

$$0 = M(0) = \lim_{t\to 0}\frac{M(t)}{t} = \lim_{t\to\infty}\frac{t}{M(t)}$$

Denote by L_M the Orlicz space on $[0,1]$ (see e.g. [14, 15, 17]) endowed with the norm

$$\|x\|_{L_M} = \inf\{\lambda : \lambda > 0, \int_0^1 M(|x(t)|/\lambda)dt \le 1\}.$$

Finally, we denote by $\chi_e(t)$ the indicator function of the measurable set $e \subset [0,\infty)$ and by $\varphi_X(t)$ the fundamental function of an r.i. space X, i.e., $\varphi_X(t) = \|\chi_e\|_X$, where $e\subset[0,1]$ and $m(e)=t$.

1.2. **The Kruglov property in r.i. spaces.** Let f be a measurable function (a random variable) on $[0,1]$. By $\pi(f)$ we denote the random variable $\sum_{i=1}^N f_i$, where f_i's are independent copies of f and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$.

Definition 1. *An r.i. space X is said to have the Kruglov property (we write: $X\in(\mathbb{K})$), if and only if $f\in X \Longleftrightarrow \pi(f)\in X$.*

This property has been studied by M. Sh. Braverman [7] which uses some probabilistic constructions of V.M. Kruglov [16] and by the first and third named authors in [2, 3, 4] via an operator approach. Note that only the implication $f\in X \Longrightarrow \pi(f)\in X$ is non-trivial, since the implication $\pi(f)\in X\Longrightarrow f\in X$ is always satisfied [7, p.11]. Moreover, an r. i. space $X\in(\mathbb{K})$ if $X\supseteq L_p$ for some $p<\infty$ [7, Theorem 1.2] and also [2, Corollaries 5.4, 5.6]. At the same time, some exponential Orlicz spaces which do not contain L_q for any $q<\infty$ also possess this property (see [7] and [2]).

2. Rearrangement-invariant spaces with $\Gamma(X)=\{1\}$ and $\Gamma_i(X)=[1,2]$.

As it was remarked in the Introduction, the Kruglov property proved to be very useful in the study of Banach-Saks type properties in r. i. spaces. The following statement from [4] (see also [3]) will play a crucial role in our considerations here.

Theorem 2.1. *If an r.i. space X on $[0,1]$ has the Kruglov property, then there exists $C>0$ such that for any sequence of independent mean zero variables $\{f_k\}_{k=1}^n\subset X$ $(n\in\mathbb{N})$ the following inequality holds:*

$$\left\|\sum_{k=1}^n f_k\right\|_X \le C\left\|\sum_{k=1}^n \bar{f}_k\right\|_{Z_X^2}. \tag{1}$$

Here, Z_X^2 is the r.i. space on $[0,\infty)$ such that the quasinorm

$$\|f\|_{Z_X^2} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_2[1,\infty)} < \infty,$$

and the sequence $\{\bar{f}_k\}_{k=1}^n \subseteq Z_X^2$ is a sequence of disjoint translates of $\{f_k\}_{k=1}^n \subseteq X$, that is, $\bar{f}_k(\cdot) = f_k(\cdot - k + 1)$. Note that inequality (1) has been proved earlier in [12] (see inequality (3) there) under the more restrictive assumption that $X \supseteq L_p$ for some $p < \infty$.

We shall also need the following result, which goes back to the proof of [19, Theorem 1]. We include details of proof for the sake of completeness and for the convenience of the reader.

Proposition 2.2. *For every r.i. space X on* $[0,1]$

$$\Big\|\sum_{k=1}^n \bar{f}_k\Big\|_{Z_X^2} \le 6\Big\|\Big(\sum_{k=1}^n f_k^2\Big)^{1/2}\Big\|_X, \tag{2}$$

where $\{f_k\}_{k=1}^n$ $(n \ge 1)$ is an arbitrary sequence of independent random variables from X.

Proof. Let $\{f_k\}_{k=1}^n \subset X$ $(n \ge 1)$ be an arbitrary sequence of independent random variables. Set

$$A(t) := \left(\sum_{k=1}^n \bar{f}_k\right)^*(t).$$

By [11, Proposition 2.1],

$$\frac{1}{2}m\{A\chi_{[0,1]} > t\} \le m\{\max_{1\le k\le n}|f_k| > t\} \le m\{A\chi_{[0,1]} > t\}, \tag{3}$$

and therefore we have, setting for brevity $B(t) := \max_{1\le k\le n}|f_k(t)|$, that

$$\frac{1}{2}\int_0^1 A(t)\,dt \le \int_0^1 B(t)\,dt \le \int_0^1 A(t)\,dt. \tag{4}$$

Let now $\{g_k\}_{k=1}^n$ be a sequence of independent identically distributed random variables such that $m\{g_k = 1\} = \frac{1}{r}$ and $m\{g_k = 0\} = \frac{r-1}{r}$ $(k \ge 1)$, where $1 < r \le n$ is a positive integer. Assume also that this sequence is independent of the sequence $\{f_k\}_{k=1}^n$. Consider now the sequence of independent random variables $\{f_k g_k\}_{k=1}^n = \{f_k(t)g_k(s)\}_{k=1}^n$, which, for convenience, we view as a sequence of random variables on the square $[0,1]\times[0,1]$. Denote $\mathcal{A} := \Big(\sum_{k=1}^n \overline{f_k g_k}\Big)^*$ and $\mathcal{B} := \max_{1\le k\le n}|f_k g_k|$ the quantities analogous to A and B (defined above for the sequence $\{f_k\}_{k=1}^n$). Applying (4) to the sequence $\{f_k g_k\}_{k=1}^n$ and $\mathcal{A}$, $\mathcal{B}$, we have

$$\frac{1}{2}\int_0^1 \mathcal{A}(u)\,du \le \int_0^1 \mathcal{B}(u)\,du \le \int_0^1 \mathcal{A}(u)\,du.$$

Observing that the function $(f_k g_k)^*$ coincides with the function $(\sigma_{\frac{1}{r}} f_k)^*$, where $\sigma_{\frac{1}{r}} f_k(t) = f_k(rt)$, we have

$$\begin{aligned}\int_0^1 \mathcal{A}(u)\,du &= \int_0^1 \left(\sum_{k=1}^n \overline{f_k g_k}\right)^*(u)\,du \\ &= \int_0^1 \left(\sum_{k=1}^n \overline{\sigma_{\frac{1}{r}} f_k}\right)^*(u)\,du = \int_0^1 \sigma_{\frac{1}{r}} A(u)\,du.\end{aligned}$$

Next, since

$$\int_0^1 \sigma_{\frac{1}{r}} Au\,du = \int_0^1 A(ru)\,du = \frac{1}{r}\left(\int_0^1 A(u)du + \int_1^r A(u)du\right)$$

and since

$$\sum_{k=2}^r A(k) \le \int_1^r A(u)\,du \le \sum_{k=1}^r A(k),$$

we rewrite the estimates above as

(5)
$$\frac{1}{4r}\left(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\right) \le \int_0^1 \mathcal{B}(u)\,du \le \frac{1}{r}\left(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\right).$$

To estimate the expression on the left hand side in (5), we denote by $\mathbb{E}(\cdot|\mathcal{M})$ the expectation operator with respect to the σ-subalgebra of the σ-algebra of all Lebesgue measurable subsets of the square $[0,1]\times[0,1]$ consisting of all sets of the form $e \times [0,1]$, where e is an arbitrary measurable subset of $[0,1]$. Clearly, for an arbitrary integrable function f on $[0,1]\times[0,1]$, we have $\mathbb{E}(f|\mathcal{M})(t) = \int_0^1 f(t,s)\,ds$. In particular,

$$\mathbb{E}(\mathcal{B}|\mathcal{M})(t) = \int_0^1 \max_{1\le k\le n} |f_k(t)g_k(s)|\,ds.$$

If now $\mathcal{A}_t$ is the quantity analogous to A defined this time for the sequence $\{f_k(t)g_k(\cdot)\}_{k=1}^n$, then

$$\int_0^1 \mathcal{A}_t(u)\,du = \int_0^1 \sum_{k=1}^r (f_k(t))^* \chi_{\left(\frac{k-1}{r},\frac{k}{r}\right)}(s)\,ds = \frac{1}{r}\sum_{k=1}^r (f_k(t))^*,$$

where $\{(f_k(t))^*\}_{k=1}^r$ is the sequence consisting of the first r entries from the sequence $\{(f_k(t))^*\}_{k=1}^n$, which is the rearrangement in the non-increasing ordering of the numbers $|f_1(t)|$, $|f_2(t)|$, ..., $|f_n(t)|$ counting multiplicities (for fixed $t \in [0,1]$). Therefore, applying (4) to the sequence $\{f_k(t)g_k(\cdot)\}_{k=1}^n$), we have

$$\frac{1}{2r}\sum_{k=1}^r (f_k(t))^* \le \mathbb{E}(\mathcal{B}|\mathcal{M})(t) \le \frac{1}{r}\sum_{k=1}^r (f_k(t))^*, \quad 0 < t \le 1. \tag{6}$$

Since $\int_0^1 \mathcal{B}(u)\,du = \int_0^1 \mathbb{E}(\mathcal{B}|\mathcal{M})(t)dt$, then we infer from (6) and (5)

$$\frac{1}{4}\left(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\right) \le \int_0^1 \sum_{k=1}^r (f_k(t))^*\,dt \le 2\left(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\right). \tag{7}$$

We may now finish the proof. Observe first that the inequality (2) follows immediately from the definition of the space Z_X^2 and from the following two estimates:

$$\|A\chi_{[0,1]}\|_X \le 2\Big\|\Big(\sum_{k=1}^n f_k^2\Big)^{1/2}\Big\|_X \tag{8}$$

and

$$\|A\chi_{[1,\infty)}\|_{L_2} \le 4\Big\|\Big(\sum_{k=1}^n f_k^2\Big)^{1/2}\|_X. \tag{9}$$

The estimate (8) follows from the left hand side inequality in (3) and the obvious inequality

$$\max_{1\le k\le n}|f_k| \le \Big(\sum_{k=1}^n f_k^2\Big)^{1/2}.$$

It remains to prove (9). First of all observe that

$$\|A\chi_{[1,\infty)}\|_{L_2} \le \Big(\sum_{k=1}^n A(k)^2\Big)^{1/2}. \tag{10}$$

Next, for brevity, denote by $\|\cdot\|_{\mathbf{s_r}}$ $(r \le n)$ the norm of the space of all scalar sequences of length n given by

$$\|x\|_{\mathbf{s_r}} = \|\{(x_k)\}_{k=1}^n\|_{\mathbf{s_r}} := \sum_{k=1}^r x_k^*.$$

We now use the observation that

$$\Big(\sum_{k=1}^n f_k^2(t)\Big)^{1/2} = \sup_{b=\{b_k\}_{k=1}^n,\ \|b\|_2\le 1} \sum_{k=1}^n (f_k(t))^* b_k^*$$
$$= \sup_{b=\{b_k\}_{k=1}^n,\ \|b\|_2\le 1} \sum_{r=1}^n \|\{(f_k(t))^*\}_{k=1}^n\|_{\mathbf{s_r}} (b_r^* - b_{r+1}^*), \quad b_{n+1}^* = 0,$$

to infer

$$\int_0^1 \Big(\sum_{k=1}^n f_k^2(t)\Big)^{1/2} dt \ge \sup_{b=\{b_k\}_{k=1}^n,\ \|b\|_2\le 1} \sum_{r=1}^n (b_r^* - b_{r+1}^*) \int_0^1 \|\{(f_k(t))^*\}_{k=1}^n\|_{\mathbf{s_r}}\, dt.$$

Using the last inequality and (7), we obtain

$$4\int_0^1 \Big(\sum_{k=1}^n f_k^2(t)\Big)^{1/2} dt \ge \sup_{b=\{b_k\}_{k=1}^n,\ \|b\|_2\le 1} \sum_{r=1}^n (b_r^* - b_{r+1}^*) \sum_{k=1}^r A(k)$$
$$= \sup_{b=\{b_k\}_{k=1}^n,\ \|b\|_2\le 1} \sum_{k=1}^n A(k) b_k^* = \Big(\sum_{k=1}^n A(k)^2\Big)^{1/2}.$$

Combining this last estimate with (10) and then applying the obvious inequality $\int_0^1 (\sum_{k=1}^n f_k^2(t))^{1/2} dt \le \|(\sum_{k=1}^n f_k^2)^{1/2}\|_X$, we deduce (9). This completes the proof. □

A Banach lattice X is said to be 2-*convex*, if there exists a constant $M > 0$ such that for every finite sequence $\{x_j\}_{j=1}^n \subseteq X$,

$$\left\|\left(\sum_{j=1}^n |x_j|^2\right)^{1/2}\right\|_X \le M\left(\sum_{j=1}^n \|x_j\|_X^2\right)^{1/2}.$$

We are now in a position to prove the following

Theorem 2.3. *If X is a 2-convex r.i. space such that $X \in (\mathbb{K})$ then $\Gamma_i(X) = [1,2]$.*

Proof. Suppose that $\{f_n\}_{n\ge 1} \subset X$ is an arbitrary sequence of mean zero independent random variables such that $f_n \to 0$ weakly and $\|f_n\|_X \le 1$, $n \ge 1$.

Combining Theorem 2.1 and Proposition 2.2, we obtain that there exists a constant $C > 0$ such that

$$\Big\|\sum_{k=1}^n f_k\Big\|_X \le C\Big\|\sum_{k=1}^n \bar{f}_k\Big\|_{Z_X^2} \le 6C\Big\|\Big(\sum_{k=1}^n f_k^2\Big)^{1/2}\Big\|_X,$$

for every $n \geq 1$. Since X is a 2-convex space, there exists a constant $C_1 > 0$ such that

$$\Big\|\Big(\sum_{k=1}^{n} f_k^2\Big)^{1/2}\Big\|_X \leq C_1\Big(\sum_{k=1}^{n} \|f_k\|_X^2\Big)^{1/2} \leq C_1\sqrt{n}, \quad n \geq 1.$$

This yields that the sequence $\{f_k\}_{k\geq 1}$ is a 2-Banach-Saks sequence.

Suppose now $\{g_n\}_{n\geq 1} \subseteq X$ is an arbitrary weakly null sequence of independent random variables, $\|g_n\|_X \leq 1$, $n \geq 1$. Setting

$$f_k = g_k - \int_0^1 g_k(t)\,dt, \quad k \geq 1,$$

we obtain a mean zero sequence of independent random variables satisfying $\|f_k\|_X \leq 2$, $k \geq 1$. Passing to a subsequence, we may assume that $|\int_0^1 g_k(t)\,dt| \leq 2^{-k}$. Then, as we have just proved, we have

$$\Big\|\sum_{k=1}^{m} g_k\Big\|_X \leq \Big\|\sum_{k=1}^{m} f_k\Big\|_X + \sum_{k=1}^{m}\Big|\int_0^1 g_k(t)\,dt\Big| \leq C_3\sqrt{m} + 1, \ \ m \geq 1.$$

This shows that $\{g_n\}_{n\geq 1}$ is a 2-Banach-Saks sequence in X and thus $\Gamma_i(X) = [1, 2]$. □

Proposition 2.4. *There exists a 2-convex reflexive r. i. space X such that $X \in (\mathbb{K})$ and $\alpha_X = 0$.*

Proof. Let $0 < \gamma < 1$ and $t_k \downarrow 0$. Let $\varphi(t)$ be a linear function on $[t_{k+1}, t_k]$ and $\varphi(t_k) = t_k^\gamma$ for every $k \in \mathbb{N}$, $\varphi(0) = 0$. Then φ is a concave function on $[0, 1]$ and $\varphi(t) \leq t^\gamma$ for each $t \in [0, 1]$. It is easy to check that

$$\liminf_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} = 1 \tag{11}$$

if t_k converges to 0 sufficiently quickly.

Consider the space $X = \Lambda(\varphi)(2)$ endowed with the norm

$$\|x\|_X = \left(\int_0^1 x^{*2}(t)d\varphi(t)\right)^{1/2}.$$

Since $\Lambda(\varphi)$ is a separable and maximal r. i. space then by Lozanovskii's theorem [18] X is reflexive. Clearly,

$$\alpha_X = \frac{1}{2}\alpha_{\Lambda(\varphi)}, \quad \beta_X = \frac{1}{2}\beta_{\Lambda(\varphi)}.$$

It is easy to prove that (11) implies $\alpha_{\Lambda(\varphi)} = 0$. Indeed, the norm of any linear operator in a Lorentz space is attained on the set of characteristic functions [15, Lemma 2.5.2]. In particular,

$$\left\|\sigma_{1/n}\right\|_{\Lambda(\varphi)} = \sup_{0<t\le\frac{1}{n}} \frac{\varphi(t)}{\varphi(nt)}, \quad n\in\mathbb{N}.$$

By [15, Lemma 2.1.3] it follows from (11) that

$$\liminf_{t\to 0} \frac{\varphi(nt)}{\varphi(t)} = 1$$

for each $n\in\mathbb{N}$. Hence $\left\|\sigma_{1/n}\right\|_{\Lambda(\varphi)} = 1$ for every $n\in\mathbb{N}$. Consequently, $\alpha_{\Lambda(\varphi)} = 0$ and $\alpha_X = 0$.

The assumption $\varphi(t)\le t^\gamma$ implies $t^{-\varepsilon}\in\Lambda(\varphi)$ for $0<\varepsilon<\gamma$. Indeed,

$$\int_0^1 t^{-\varepsilon} d\varphi(t) = \varphi(t)t^{-\varepsilon}\Big|_0^1 + \varepsilon\int_0^1 \varphi(t)t^{-\varepsilon-1}dt \le$$
$$\le \varphi(1) + \varepsilon\int_0^1 t^{\gamma-\varepsilon-1}dt = \varphi(1) + \frac{\varepsilon}{\gamma-\varepsilon} < \infty.$$

Hence $t^{-\varepsilon/2}\in X$ and $L_{\frac{2}{\varepsilon}}\subset X$. Using [7, Theorem 1.2], we conclude that $X\in(\mathbb{K})$. □

The condition $\Gamma(X)\ne\{1\}$ guarantees that $\alpha_X>0$ (see [1, Theorem 4.2(i)]), therefore combining Theorem 2.3 and Proposition 2.4 yields the following theorem.

Theorem 2.5. *There exists a reflexive r. i. space X with minimal $\Gamma(X)$ and maximal $\Gamma_i(X)$, that is $\Gamma(X)=\{1\}$ and $\Gamma_i(X)=[1,2]$.*

Remark. Let φ be the same function as in the proof of Proposition 2.4. Then the space $X=\Lambda(\varphi)(p)$, equipped with the norm $\|x\|_X = \left(\int_0^1 x^{*p}(t)d\varphi(t)\right)^{1/p}$ is a reflexive r. i. space such that $\Gamma(X)=\{1\}$ and $\Gamma_i(X)=[1,p]$ if $1<p\le 2$.

Remark. The assumption $\Gamma_i(X)=[1,2]$ does not imply even the BS-property of X. Let $L^0_{\exp}$ be the separable part of the Orlicz space $L_{\exp}(0,1)$ generated by the function e^t-1. Then $L^0_{\exp}\in(\mathbb{K})$ [7, p. 42] (see also [16, 2]). The fact that $L^0_{\exp}$ is 2-convex follows from the fact that the function e^t-1 is 2-convex. Finally, the space $L^0_{\exp}$ does not have the Banach-Saks property [10, Theorem 5.5(ii)] (in particular, $\Gamma(L^0_{\exp})=\{1\}$).

We complete this section with the following result, which is based on the study of symmetric structure of r. i. spaces with Kruglov property from [4] and which strengthens Theorem 4.3 from [1].

Theorem 2.6. *If X is an r. i. space such that $X \in (\mathbb{K})$, then the following conditions are equivalent*

(i). $X \in (BS)$ *(respectively,* $X \in BS(p)$*);*
(ii). $Z_X^2 \in (BS)$ *(respectively,* $Z_X^2 \in BS(p)$*).*

Proof. It follows from [4, Theorem 4.1] that the assumption $X \in (\mathbb{K})$ guarantees that X contains a subspace isomorphic to the space Z_X^2. The fact that the space Z_X^2 contains a subspace isomorphic to X holds for any r. i. space X. □

Recall that a Banach lattice X is said to have the Banach-Saks d-property ($X \in (dBS)$) if each weakly null sequence of pairwise disjoint elements from X contains a BS-sequence.

Corollary 2.7. *If X is an r. i. space such that $X \in (\mathbb{K})$ and X has the Fatou property, then the following conditions are equivalent*

(i). $X \in (BS)$*;*
(ii). $Z_X^2 \in (BS)$*;*
(iii). $Z_X^2 \in (dBS)$.

Proof. The fact that Z_X^2 has the Fatou property follows from [1, Lemma 3.3]. Therefore, the equivalence between (ii) and (iii) is obtained by referring to [10, Theorem 4.5]. Finally, using Theorem 2.6 we obtain the result. □

3. Spaces with fundamental function $t^{1/2}$ and index 2

The following statement is essentially known (see e.g. [8]). We present a simple proof for the sake of completeness.

Lemma 3.1. *Let x be a measurable function on $[0,1]$, $0 \le x(t) \le 1$ for each $t \in [0,1]$ and $\lambda = \int_0^1 x(t)dt$. There exists a sequence of measurable sets $e_n \subset [0,1]$ such that $m(e_n) = \lambda$ for any $n \in \mathbb{N}$ and χ_{e_n} tends to x weakly in L_1.*

Proof. We first consider the case $x(t) = a\chi_{(0,b)}(t)$ where $ab = \lambda$ and $0 < a, b < 1$. We put

$$e_n = \bigcup_{k=0}^{2^n-1} (k2^{-n}b, k2^{-n}b + \lambda 2^{-n}) = \bigcup_{k=0}^{2^n-1} (k2^{-n}b, (k+a)2^{-n}b),\ n \in \mathbb{N},$$

and remark that $\chi_{e_n}(t) = z(2^n t)$, $t \in (0,1)$, where $z(t)$ is the b-periodic function given by

$$z(t) = \sum_{k\in\mathbb{Z}} \chi_{[kb,(k+a)b]}(t) \ \ (t \in \mathbb{R}).$$

Since $e_n \subset [0,b]$, then by Fejér's Lemma ([6, §1.20] or [9, Ex. 2.16])

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 y(t)\chi_{e_n}(t)\,dt &= \lim_{n\to\infty}\int_0^b y(t)\chi_{e_n}(t)\,dt \\ &= \frac{1}{b}\int_0^b y(t)dt\int_0^b z(t)\,dt \\ &= a\int_0^b y(t)\,dt = \int_0^1 y(t)x(t)\,dt\end{aligned}$$

for any $y \in L_1$.

Let

$$x(t) = \sum_{k=1}^m a_k\chi_{(b_{k-1},b_k)}(t),$$

where $0 = b_0 < b_1 < \ldots < b_m = 1$. Applying the construction described above to every interval (b_{k-1}, b_k), $1 \le k \le m$, we get a sequence of sets e_n $(n \in \mathbb{N})$ such that

$$\lim_{n\to\infty}\int_0^1 y(t)\chi_{e_n}(t)\,dt = \int_0^1 y(t)x(t)\,dt$$

for every $y \in L_1$.

It is easy to extend the statement proved above from the set of step functions to the whole of L_1. We omit this part of the proof because we shall use our statement later on for step functions only. □

Theorem 3.2. *L_2 is the unique r. i. space with index 2 and containing $L_{2,1}$.*

Proof. Let X be an r. i. space such that $\gamma(X) = 2$ and $L_{2,1} \subset X$. We can suppose that $\|y\|_X \le \|y\|_{L_{2,1}}$ for every $y \in L_{2,1}$. In particular,

$$\|\chi_e\|_X \le \|\chi_e\|_{L_{2,1}} = m(e)^{1/2} \tag{12}$$

for any $e \subset [0,1]$. It is explained in the Introduction of [20] that X is separable. Moreover, by [20, Theorem 4.8], $X \subset L_2$. Let x be a step function and $0 \le x(t) \le 1$ for every $t \in [0,1]$. Denote $\lambda = \|x\|_{L_2}$. Then $\int_0^1 x^2(t)dt = \lambda^2$. By Lemma 3.1, there exists a sequence $e_n \subset [0,1]$ such that $m(e_n) = \lambda^2$ for every $n \in \mathbb{N}$ and χ_{e_n} tends to x^2 weakly in L_1.

Consider the sequence $\{r_k(s)\chi_{e_k}(t)\}_{k\ge1}$, where $\{r_k\}_{k\ge1}$ is a sequence of Rademacher functions on $[0,1]$ (that is a sequence of independent identically distributed centered Bernoulli variables on $[0,1]$). The sequence $\{r_i \otimes \chi_{e_i}\}_{i\ge1}$ tends to 0 weakly in $X([0,1]\times[0,1])$ [20, Lemma 3.4]. Since by the assumption the space X has the 2-BS-property, we have

$$\Big\|\sum_{i=1}^{j(k)} r_{n(i)} \otimes \chi_{e_{n(i)}}\Big\|_{X([0,1]\times[0,1])} \le Cj^{1/2}(k)\|\chi_{e_1}\|_X \tag{13}$$

for every $k \in \mathbb{N}$ and some subsequence $\{n(i)\} \subset \mathbb{N}$, where a constant $C > 0$ depends on X only [20, Lemma 4.2]. Furthermore, the estimate (13) holds for any subsequence of $\{n(i)\}$.

Fix a sequence $\{n(k)\} \subset \mathbb{N}$ satisfying (13). By [10, Proposition 4.3] there exists a subsequence $\{n'(k)\} \subset \{n(k)\}$ and a function $z \in L_1$ such that the sequence $\{\frac{1}{j}\sum_{k=1}^{j} \chi_{e_{m(k)}}\}$ tends to z in measure for any subsequence $\{m(k)\} \subset \{n'(k)\}$.

Consider now the sequence $\{y_j\}_{j\ge1} := \{\frac{1}{j}\sum_{k=1}^{j} \chi_{e_{n'(k)}}\}_{j\ge1}$ which clearly tends to x^2 weakly in L_1. Since $\{y_j\}_{j\ge1}$ tends to z in measure, we have $x^2 = z$. Consequently, $\{y_j\}_{j\ge1}$ tends to x^2 in measure and some subsequence $\{y_{j(k)}\}$ tends to x^2 a. e. Hence $\{y_{j(k)}^{1/2}\}$ tends to x a. e. Since X is separable and $0 \le y_{j(k)}^{1/2}(t) \le 1$ for every $k \in \mathbb{N}$ and every $t \in [0,1]$, we deduce that $\{y_{j(k)}^{1/2}\}$ tends to x in X and

$$\|x\|_X = \lim_{k\to\infty} \|y_{j(k)}^{1/2}\|_X. \tag{14}$$

It can easily be checked that the sequence $\{r_i \otimes \chi_{e_i}\}_{i\ge1}$ is unconditional in the space $X([0,1]\times[0,1])$ with unconditional constant equal to 1. Therefore, by [17, 1.d.6]

$$\|y_{j(k)}^{1/2}\|_X \le \sqrt{2}\left\|\frac{1}{j^{1/2}(k)}\sum_{i=1}^{j(k)} r_i \otimes \chi_{e_i}\right\|_{X([0,1]\times[0,1])} \tag{15}$$

for every $k \in \mathbb{N}$ and every sequence $\{j(k)\}_{k\ge1}$.

It follows from (12)—(15) that

$$\|x\|_X \le C\sqrt{2}\lambda = C\sqrt{2}\|x\|_{L_2}.$$

Clearly, the last inequality can be extended to all step functions. Since the set of step functions is dense in X and L_2, the estimate above holds for every $x \in L_2$. This proves that $X = L_2$ with equivalence of norms. □

Theorem 4.8 from [20] states that $X \subset L_2$ if $\gamma(X) = 2$. This statement is exact in the following sense.

Proposition 3.3. *Let $z \in L_2$. There exists an r. i. space X such that $z \in X \subsetneq L_2$ and $\gamma(X) = 2$.*

PROOF. Note that $z^2 \in L_1$. By [14, § 8] there exists a convex increasing function $M(u)$ on $[0,\infty)$ such that $M(0)=0$, $\sup_{u>0}\frac{M(2u)}{M(u)}<\infty$, $\lim_{u\to\infty}\frac{M(u)}{u}=\infty$ and $M(z^2)\in L_1$. Then the Orlicz space L_M endowed with the norm

$$\|x\|_{L_M}=\inf\left\{\lambda>0:\ \int_0^1 M(|x(t)|/\lambda)\,dt\ \le 1\right\}$$

is a separable r. i. space and $\alpha_{L_M}>0$ [17, Proposition 2.b.5]. Consider the space X endowed with the norm

$$\|x\|_X=\|x^2\|_{L_M}^{1/2}.$$

It is easy to check that $\alpha_X=\frac{1}{2}\alpha_{L_M}$. Hence $\alpha_X>0$. Clearly, $z\in X$, X is 2-convex and $X\neq L_2$. By [20, Theorem 4.3], we have: $\gamma(X)=2$. □

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