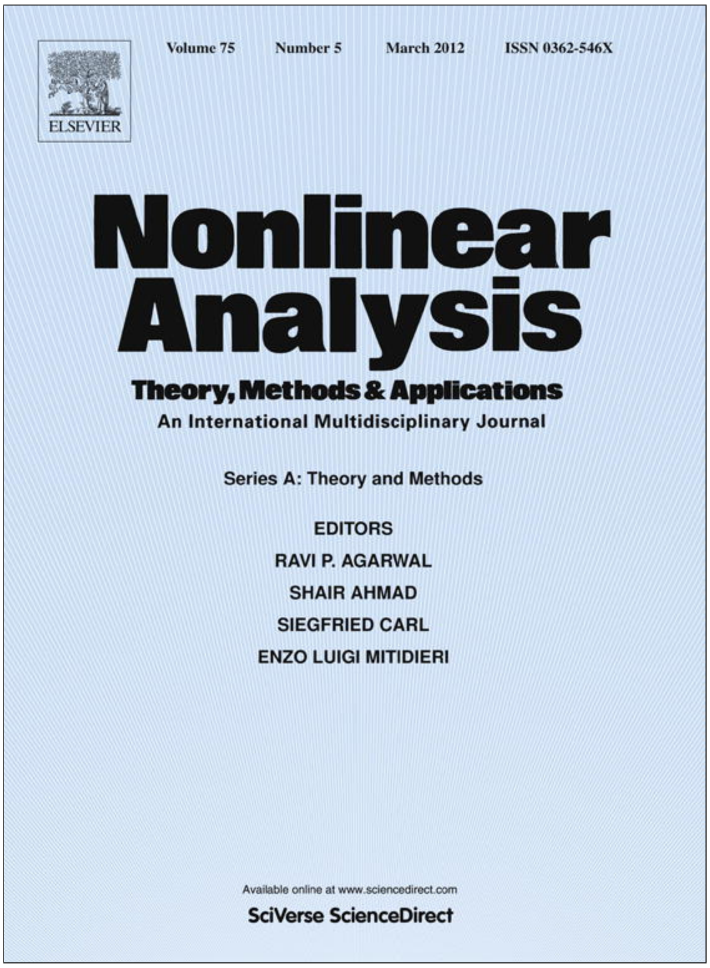
Volume 75 Number 5 March 2012 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
EDITORS
RAVI P. AGARWAL
SHAIR AHMAD
SIEGFRIED CARL
ENZO LUIGI MITIDIERI
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Nonlinear Analysis 75 (2012) 2735–2749

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On extrapolation of rearrangement invariant spaces

S.V. Astashkin [a], K.V. Lykov [a], M. Mastyło [b,*]

[a] Samara State University, Pavlova 1, Samara 443011, Russia

[b] Faculty of Mathematics and Computer Science, Adam Mickiewicz University and Institute of Mathematics, Polish Academy of Science (Poznań branch), Umultowska 87, 61-614 Poznań, Poland

ARTICLE INFO

Article history:
Received 16 July 2011
Accepted 19 November 2011
Communicated by S. Ahmad

MSC:
46B70
46B30
46M35

Keywords:
Extrapolation spaces
Interpolation spaces
Interpolation functor
Rearrangement invariant spaces
Marcinkiewicz spaces
Orlicz spaces

ABSTRACT

We study extrapolation spaces generated by the Banach scales of L_p and $L_{p,\infty}$-spaces defined on [0, 1]. We give new extrapolatory descriptions for the Marcinkiewicz and the Orlicz–Lorentz spaces. Using interpolation of sublinear positive operators, we also prove interpolation embeddings for extrapolation spaces.

1. Introduction

The extrapolation theory has its origin in the classical Yano theorem [1]. It plays an important role in the study of operators between quasi-Banach spaces. One of the most fundamental problems in the theory of extrapolation is to characterize classes of certain quasi-Banach spaces via special constructions called extrapolation methods. The first systematic study of abstract methods of extrapolation in Banach spaces was initiated by Jawerth and Milman in their memoir [2] and it was continued by Milman in his interesting book [3]. In this book, applications in many areas of classical analysis including the theory of Fourier series, the theory of logarithmic Sobolev inequalities, commutator inequalities and abstract parabolic equations are presented. Applications of the extrapolation theory to the Fourier analysis are also given by Carro in [4] and Carro et al. in [5].

In recent years many extrapolation results were proved for some important classes of Banach function spaces called rearrangement invariant spaces. These spaces are generated by methods which involve the scale of L_p-spaces. This is motivated by applications in harmonic analysis to many important operators including the Hardy–Littlewood maximal operator and the Hilbert transform. Basic extrapolation theory is relatively a large subject, and we will not attempt to discuss carefully the topic. It should be mentioned that a full bibliography related to extrapolation spaces and their applications is not composed yet and could be long. We refer to the mentioned memoir [2] and the book [3], a survey article [6], papers [7–9] and references included therein.

In the current paper, we study extrapolation descriptions for Marcinkiewicz and Orlicz–Lorentz spaces, which play an important role in the theory of interpolation between rearrangement invariant spaces (see [10,11]). The new ingredient

* Corresponding author.
E-mail addresses: astashkin@ssu.samara.ru (S.V. Astashkin), lykich@list.ru (K.V. Lykov), mastylo@amu.edu.pl (M. Mastyło).

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.11.016

in this presentation is the combination of interpolation theory with the modern approach based on new variants of extrapolation constructions. This approach allows us both to prove new extrapolation formulas for the norms of Marcinkiewicz and Orlicz–Lorentz spaces and to estimate boundaries of their extrapolation description. Note that many results of our paper cannot be obtained by the well-known Jawerth–Milman extrapolation theory developed in [2,3] (see Section 3 of this paper and a survey article [6]).

To describe results of our paper, we need some notation and definitions.

A *Banach lattice* on a measure space (Ω, Σ, μ) is defined to be a Banach space X which is a subspace of $L^0 = L^0(\mu)$ (the topological linear space of all equivalence classes of the real Lebesgue measurable functions equipped with the topology of convergence in measure on μ-finite sets) such that there exists $u \in X$ with $u > 0$ a.e. and if $|x| \le |y|$ a.e., where $y \in X$ and $x \in L^0$, then $x \in X$ and $\|x\|_X \le \|y\|_X$.

Throughout the paper, I denotes the unit interval $[0, 1]$ equipped with the Lebesgue measure m. Given $f \in L^0(I) := L^0(I, m)$, its distribution function is defined by $m_f(\lambda) = m(\{x \in I: |f(x)| > \lambda\})$ $(\lambda > 0)$, and its decreasing rearrangement by $f^*(t) = \inf\{\lambda \ge 0: m_f(\lambda) \le t\}$ for $t > 0$. A Banach lattice $(X, \|\cdot\|_X)$ on (I, m) is called a rearrangement invariant (r.i.) space on I provided $f \in X$ and $g \in L^0(I)$ with $m_f = m_g$ implies $g \in X$ and $\|f\|_X = \|g\|_X$.

If X is a Banach r.i. space on I and χ_A denotes the characteristic function of a measurable set A, clearly, $\|\chi_A\|_X$ depends only on $m(A)$. The function $\varphi_X(t) = \|\chi_A\|_X$, where $m(A) = t$, $0 \le t \le 1$, is called the *fundamental function* of X.

Most important examples of r.i. spaces are L_p-spaces with $1 \le p \le \infty$, Marcinkiewicz and Lorentz spaces. Let $\varphi: I \to I$ be a quasi-concave function, that is, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t \in (0, 1]$ and both φ and $\widetilde{\varphi}(t) := t/\varphi(t)$ are non-decreasing functions.

The Marcinkiewicz space $M(\varphi)$ is defined to be the r.i. space of all $f \in L^0(I)$ such that

$$\|f\|_{M(\varphi)} := \sup_{0<s\le 1} \frac{\varphi(s)}{s} \int_0^s f^*(t)\,dt < \infty.$$

If $\varphi: I \to I$ is an increasing concave function, $\varphi(0) = 0$, the Lorentz space $\Lambda(\varphi)$ consists of all $f \in L^0(I)$ such that

$$\|f\|_{\Lambda(\varphi)} := \int_0^1 f^*(t)\,d\varphi(t) < \infty.$$

If $\varphi(t) = t^{1/p}$ for all $t \in I$ $(1 \le p < \infty)$, then the Marcinkiewicz space $M(\varphi)$ will be denoted by $L_{p,\infty}$ and we put $\|\cdot\|_{p,\infty} := \|\cdot\|_{L_{p,\infty}}$. Clearly, $L_{1,\infty} = L_1$ with equality of norms. The fundamental functions of these spaces are $\varphi_{\Lambda(\varphi)} = \varphi_{M(\varphi)} = \varphi$.

Let $M: [0, \infty) \to [0, \infty)$ be an Orlicz function (i.e., M is a continuous non-decreasing convex function on $(0, \infty)$ with $M(0) = 0$) and φ be a concave function on I. The Orlicz–Lorentz space $\Lambda_M(\varphi)$ consists of all $f \in L^0(I)$ for which there exists $\lambda > 0$ such that

$$\int_0^1 M\left(f^*(t)/\lambda\right)\,d\varphi(t) < \infty.$$

Then $\Lambda_M(\varphi)$ is an r.i. space on I equipped with the norm

$$\|f\| := \inf\left\{\lambda > 0: \int_0^1 M\left(f^*(t)/\lambda\right)\,d\varphi(t) \le 1\right\}.$$

If $\varphi(t) = t$ for all $t \in [0, 1]$, then $\Lambda_M(\varphi)$ is the well-known Orlicz space L_M with the Luxemburg norm.

We note that L^1 and L^∞ are, respectively, the largest and the smallest r.i. spaces on I. Moreover, if X is an r.i. space on I with the fundamental function φ, then φ is quasi-concave and the following continuous embeddings hold (see [11, Theorems II.5.5 and II.5.7] or [10, Theorem II.5.13]):

$$\Lambda(\bar{\varphi}) \hookrightarrow X \hookrightarrow M(\varphi),$$

where $\bar{\varphi}$ is the least concave majorant of φ.

For a given $t > 0$, the *dilation operator* σ_t is defined by

$$\sigma_t x(s) = x(s/t)\chi_{[0,1]}(s/t), \quad s \in [0, 1].$$

Then σ_t is bounded in every r.i. space X and so the *lower* and the *upper Boyd indices* α_X and β_X are well-defined, where

$$\alpha_X = \lim_{t\to 0+} \frac{\log \|\sigma_t\|_{X\to X}}{\log t}, \qquad \beta_X = \lim_{t\to\infty} \frac{\log \|\sigma_t\|_{X\to X}}{\log t},$$

respectively. In general, we have $0 \le \alpha_X \le \beta_X \le 1$.

The *lower* and *upper dilation indices* of a quasi-concave function $\psi: [0, 1] \to [0, \infty)$ are defined by

$$\gamma_\psi = \lim_{t\to 0+} \frac{\log \mathcal{M}_\psi(t)}{\log t} \quad \text{and} \quad \delta_\psi = \lim_{t\to\infty} \frac{\log \mathcal{M}_\psi(t)}{\log t},$$

respectively, where $\mathcal{M}_\psi(t) = \sup_{0<s\le 1, 0<st\le 1} \psi(st)/\psi(s)$. We have $0 \le \gamma_\psi \le \delta_\psi \le 1$, and since $\mathcal{M}_{\varphi_X}(t) \le \|\sigma_t\|_{X\to X}$ for every $t > 0$, it follows that $\alpha_X \le p_X \le q_X \le \beta_X$, where $p_X := \gamma_{\varphi_X}$ and $q_X := \delta_{\varphi_X}$. For general properties of r.i. spaces and indices we refer to [10,11].

Let $\mathcal{X}$ be a real vector space and let $S: \mathcal{X} \to L^0(\mu)$ be a sublinear operator, i.e., satisfying the conditions: (1) $S(x+y) \le Sx + Sy$ and (2) $S(\lambda x) = |\lambda| S(x)$ for all $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{R}$. Moreover, suppose that (3) $Sx = 0$ if and only if $x = 0$. For a given Banach lattice $(F, \|\cdot\|_F)$ on (Ω, μ) and for such an operator $S: \mathcal{X} \to L^0(\mu)$, we define a linear space

$$D_F(S) := \{x \in \mathcal{X}: Sx \in F\}$$

equipped with the norm

$$\|x\|_{D_F(S)} = \|Sx\|_F, \quad x \in D_F(S).$$

For a study of Banach spaces generated via sublinear operators we refer to [12].

Further, we will be interested in a special case of spaces generated via sublinear operators which is important for applications to extrapolation.

Let $\mathcal{X}$ be a vector space, and let $\psi: \Omega \to (0, \infty)$ be a positive function. Assume that $\left(\|\cdot\|_{\psi(\omega)}\right)_{\omega\in\Omega}$ is a family of norms defined on a vector space $\mathcal{X}$ such that $\|x\|_{\psi(\cdot)} \in L^0(\mu)$ for all $x \in \mathcal{X}$. Clearly, the operator $S_\psi: \mathcal{X} \to L^0(\mu)$ defined for all $x \in \mathcal{X}$ by

$$S_\psi x(\omega) = \|x\|_{\psi(\omega)}, \quad \omega \in \Omega$$

satisfies the above properties (1)–(3). In what follows, the space $D_F(S_\psi)$ is denoted by $D_F(\psi)$.

If $\mathcal{X} = \bigcap_{1\le p<\infty} L_p(I)$ and F is a Banach lattice on $([1,\infty), m)$, then the space $D_F(\psi)$ generated by the norms $\|x\|_{\psi(p)} := \|x\|_{L_p(I)}$ for all $1 \le p < \infty$ is denoted by $\mathcal{L}_F$. It easy to check that $\mathcal{L}_F$ is an r.i. space on I such that, for all $1 \le p < \infty$,

$$\mathcal{L}_F \hookrightarrow L_p.$$

The lattice properties of the Banach spaces $\mathcal{L}_F$ were studied in [13], where some applications to convergence of orthogonal series are presented.

Following [8], an r.i. space X on I is said to be an *extrapolation space* if there exists a Banach lattice F on $([1,\infty), m)$ such that $X = \mathcal{L}_F$. We denote the set of all extrapolation spaces by $\mathcal{E}$.

We briefly describe the content of our paper. In Section 2, we introduce the class $\mathcal{E}'$ of r.i. spaces generated by the scale of Marcinkiewicz spaces $L_{p,\infty}$ $(1 \le p < \infty)$ and give a criteria for a Marcinkiewicz space $M(\varphi)$ to belong to the class $\mathcal{E}'$. The main result here is a new description of extrapolation Marcinkiewicz spaces $M(\varphi)$ under a mild assumption on φ. We also present an example which shows that there exist Marcinkiewicz spaces $M(\varphi) \in \mathcal{E}'$ such that $M(\varphi) \notin \mathcal{E}$. In Section 3, we consider a special type of extrapolation Marcinkiewicz spaces. We prove that a Marcinkiewicz space $M(\psi)$ is an extrapolation space whenever $\psi(t) = \sup_{p\ge 1} \varphi(e^{e^{-p}})\, t^{1/p}$ for all $t \in I$ for some increasing function φ on I with the property $\varphi(et) \le C\varphi(t)$ for every $0 < t \le 1/e$. We also show that there exist extrapolation Orlicz spaces which do not coincide with any Marcinkiewicz space.

In Section 4, we apply the well-known Calderón–Lozanovskii interpolation construction to obtain an extrapolation description of the Orlicz–Lorentz spaces. In Section 5, we prove some interpolation embedding formulas for spaces generated by positive interpolation functors.

Throughout the paper, we use the following notation: given two Banach spaces X and Y, we use the notion of (continuous) embedding $X \hookrightarrow Y$, which means that $X \subset Y$ and the natural inclusion map from X into Y is continuous. If f and g are real functions, then the symbol $f \asymp g$ means that $c^{-1}g \le f \le cg$ for some $c > 0$.

2. Extrapolation Marcinkiewicz spaces and the class $\mathcal{E}'$

The main aim of this section is to give a new criterion for a Marcinkiewicz space to be an extrapolation space using the class $\mathcal{E}'$ of r.i. spaces constructed by the scale of Marcinkiewicz spaces $L_{p,\infty}$, $1 \le p < \infty$.

We start with necessary definitions. If F is a Banach lattice on $[1,\infty)$, then the space $\mathcal{U}_F$ can be defined by the norms $\|x\|_{L_{p,\infty}(I)}$ in the same way as the space $\mathcal{L}_F$. Namely, by $\mathcal{U}_F$ we will denote the set of all $x \in L^0(I)$ such that $x \in L_p(I)$ for any $1 \le p < \infty$ and the function $\eta_x(p) := \|x\|_{p,\infty} \in F$. It is an r.i. space endowed with the norm $\|x\|_{\mathcal{U}_F} := \|\eta_x\|_F$. By $\mathcal{E}'$ we denote the class of all r.i. spaces X such that $X = \mathcal{U}_F$ with some Banach lattice F.

In the sequel we will need some auxiliary results, which we will use without any references. Let φ be a quasi-concave function on I and, as above, $\widetilde{\varphi}(t) := t/\varphi(t)$. Since $\lim_{t\to 0+} \widetilde{\varphi}(t) > 0$ implies $M(\varphi) = L_1$, we assume throughout this section that $\widetilde{\varphi}(0+) := \lim_{t\to 0+} \widetilde{\varphi}(t) = 0$. This implies that $\widetilde{\varphi}$ is absolutely continuous and so $\widetilde{\varphi}$ is differentiable for almost all $t \in [0,1]$. In what follows, we denote by $\widetilde{\varphi}'$ the function $t \mapsto (\widetilde{\varphi})'(t)$, when $(\widetilde{\varphi})'(t)$ exists and $(\widetilde{\varphi})'(t) = 0$ otherwise. Then from the definition of the norm in $M(\varphi)$ it follows that for all $x \in M(\varphi)$ we have:

$$\int_0^s x^*(t)\,dt \le \|x\|_{M(\varphi)} \int_0^s \widetilde{\varphi}'(t)\,dt, \quad 0 < s \le 1.$$

Since $L_{p,\infty}$ is an exact interpolation space between L_1 and L_∞ (see, e.g. [11, Theorems 2.4.3 and 2.4.9]) for every $1 \le p < \infty$, the condition $\widetilde{\varphi}' \in L_{p,\infty}$ $(1 \le p < \infty)$ (or equivalently, $M(\varphi) \subset L_{p,\infty}$, $1 \le p < \infty$) implies that

$$\sup_{p\ge 1} \frac{\|x\|_{p,\infty}}{\|\widetilde{\varphi}'\|_{p,\infty}} \le \|x\|_{M(\varphi)}.$$

Therefore,

$$M(\varphi) \hookrightarrow \mathcal{U}_{L_\infty(w)}, \tag{1}$$

where $w(p) = 1/\|\widetilde{\varphi}'\|_{p,\infty}$ for every $1 \le p < \infty$.

The following result is similar to Theorem 1 from [8].

Theorem 2.1. *Let φ be an arbitrary quasi-concave function on I such that $\widetilde{\varphi}' \in L_{p,\infty}$ for all $1 \le p < \infty$. Then the following conditions are equivalent.*

(i) $M(\varphi) \in \mathcal{E}'$.
(ii) $M(\varphi) = \mathcal{U}_{L_\infty(w)}$, *where* $w(p) = 1/\|\widetilde{\varphi}'\|_{p,\infty}$, $1 \le p < \infty$.
(iii) *There exists a constant $C > 0$ such that*

$$\varphi(t) \le C \sup_{p\ge 1} \frac{t^{1/p}}{\|\widetilde{\varphi}'\|_{p,\infty}}, \quad t \in I.$$

(iv) *There exists a function $\omega\colon [1,\infty) \to (0,\infty)$ such that*

$$\varphi(t) \asymp \sup_{p\ge 1} \omega(p)\, t^{1/p}, \quad t \in I.$$

(v) *We have that*

$$\varphi(t) \asymp \sup_{p\ge 1} \left(\inf_{v\in(0,1]} \frac{\varphi(v)}{v^{1/p}} \right) t^{1/p}, \quad t \in I.$$

Proof. (i) $\Rightarrow$ (ii). Assume that there exists a Banach lattice F on $[1,\infty)$ such that $M(\varphi) = \mathcal{U}_F$. Since $\widetilde{\varphi}' \in M(\varphi)$, then the function $f(p) := \|\widetilde{\varphi}'\|_{p,\infty}$ $(p \ge 1)$ belongs to F. For any $g \in L_\infty(w)$, we have that $|g(p)| \le \|g\|_{L_\infty(w)} f(p)$ a.e. on $[1,\infty)$. Hence, $g \in F$ and $\|g\|_F \le \|g\|_{L_\infty(w)} \|f\|_F$. Therefore, $L_\infty(w) \hookrightarrow F$ and so

$$\mathcal{U}_{L_\infty(w)} \hookrightarrow \mathcal{U}_F = M(\varphi).$$

Since converse embedding (1) holds, we obtain the required equality $M(\varphi) = \mathcal{U}_{L_\infty(w)}$.

(ii) $\Leftrightarrow$ (iii). The inequality from condition (iii), up to constants, is an inequality between the fundamental functions of the r.i. spaces $M(\varphi)$ and $\mathcal{U}_{L_\infty(w)}$. Thus, because of the Marcinkiewicz space being the biggest one among r.i. spaces with the same fundamental function [11, Theorem 2.5.7], the mentioned inequality is equivalent to the continuous embedding $\mathcal{U}_{L_\infty(w)} \hookrightarrow M(\varphi)$. Combining it with embedding (1) we obtain the equivalence (ii) $\Leftrightarrow$ (iii).

The implications (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (v). By the equivalence from the condition (iv), there exists a constant $C > 0$ such that for all $p \ge 1$ we have that

$$\omega(p) \le C \inf_{v\in(0,1]} \frac{\varphi(v)}{v^{1/p}}.$$

Hence, it follows that

$$\begin{aligned} \varphi(t) \asymp \sup_{p\ge 1} \omega(p)\, t^{1/p} &\le C \sup_{p>1} \left(\inf_{v\in(0,1]} \frac{\varphi(v)}{v^{1/p}} \right) t^{1/p} \\ &\le C \sup_{p\ge 1} \frac{\varphi(t)}{t^{1/p}}\, t^{1/p} = C\,\varphi(t), \end{aligned}$$

which proves the equivalence given in condition (v).

(v) $\Rightarrow$ (iii). The equivalence given in (v) implies that there exists a constant $C > 0$ such that for any $t \in (0,1]$ there exists $p \ge 1$ such that the following inequality holds:

$$\sup_{v\in(0,1]} \frac{v^{1/p}}{\varphi(v)} \le C\, \frac{t^{1/p}}{\varphi(t)}.$$

It is easy to see that the left-hand side of the last inequality is equal to $\|\widetilde{\varphi}'\|_{p,\infty}$. In consequence, for every $t \in (0,1]$ we have that

$$\varphi(t) \le C \sup_{p\ge 1} \frac{t^{1/p}}{\|\widetilde{\varphi}'\|_{p,\infty}},$$

i.e., the condition (iii) holds. $\square$

Remark 2.2. We note that Theorem 1 in [8] states that $M(\varphi) \in \mathcal{E}$ if and only if there exists a constant $C > 0$ such that

$$\varphi(t) \le C \sup_{p\ge 1} \frac{t^{1/p}}{\|\widetilde{\varphi}'\|_p}, \quad 0 < t \le 1.$$

Combining this result with Theorem 2.1 and with the obvious inequality $\left\|\widetilde{\varphi}'\right\|_{p,\infty} \le \|\widetilde{\varphi}'\|_p$ for all $p \ge 1$, we conclude that $M(\varphi) \in \mathcal{E}$ implies $M(\varphi) \in \mathcal{E}'$. We will see later that the converse implication is not true in general.

Further, we will need a technical proposition similar to Lemma 2.1 from [13]. We include a proof for the sake of completeness.

Proposition 2.3. *If an r.i. space $X \in \mathcal{E}'$, then the upper Boyd index $\beta_X = 0$.*

Proof. By the assumption, $X = \mathcal{U}_F$ for some Banach lattice F on $[1, \infty)$. Let $p_0 \in (1, \infty)$ be fixed. Define the Banach lattice $F_0 := \{f\chi_{[p_0,\infty)} : f \in F\}$ equipped with the norm $\|f\|_{F_0} := \|f\chi_{[p_0,\infty)}\|_F$. It is easy to see that $\mathcal{U}_{F_0} = \mathcal{U}_F$ with equivalence of norms. Hence, X coincides with the r.i. space $X_0 := \mathcal{U}_{F_0}$, which implies that $\beta_X = \beta_{X_0}$. Since for every $x \in L_{p,\infty}$ and $\tau > 1 \|\sigma_\tau x\|_{p,\infty} \le \tau^{1/p}\|x\|_{p,\infty}$ and $\|x\|_{X_0} = \| \|x\|_{p,\infty} \cdot \chi_{[p_0,\infty)}\|_F$, we have: $\|\sigma_\tau x\|_{X_0} \le \tau^{1/p_0}\|x\|_{X_0}$. Therefore, $\|\sigma_\tau\|_{X_0\to X_0} \le \tau^{1/p_0}$, whence $\beta_X = \beta_{X_0} \le 1/p_0$. Since p_0 can be chosen arbitrarily large, the proof is complete. □

Now, we state and prove the main result of this section. Under a mild assumption on a quasi-concave function φ, we give necessary and sufficient conditions which ensure that the Marcinkiewicz space $M(\varphi) \in \mathcal{E}$.

Theorem 2.4. *Let φ be a quasi-concave function on I with $\varphi(0+) = 0$ and such that*

$$\varphi(t) \asymp \sup_{p\ge 1} \omega(p)\, t^{1/p}$$

for some weighted function $\omega: [1, \infty) \to (0, \infty)$. The following conditions are equivalent.

(i) $M(\varphi) \in \mathcal{E}$.

(ii) $\mathcal{L}_{L_\infty(w_1)} = \mathcal{U}_{L_\infty(w_2)}$, *where* $w_1(p) = 1/\|\widetilde{\varphi}'\|_p$ *and* $w_2(p) = 1/\|\widetilde{\varphi}'\|_{p,\infty}$, $1 \le p < \infty$.

(iii)

$$\sup_{p\ge 1} \frac{t^{1/p}}{\|1/\varphi\|_p} \asymp \sup_{p\ge 1} \frac{t^{1/p}}{\|1/\varphi\|_{p,\infty}}, \quad 0 < t \le 1.$$

(iv) *There exists a constant $C > 0$ such that*

$$\|1/\varphi\|_p \le C\, \|1/\varphi\|_{p,\infty}, \quad 1 \le p < \infty.$$

Proof. We first observe that from our hypothesis on φ and from Theorem 2.1 it follows that $M(\varphi) = \mathcal{U}_{L_\infty(w_2)}$. Therefore, by Proposition 2.3, $\delta_\varphi = \beta_{M(\varphi)} = 0$. Combining this with [11, Lemma 2.1.4] we conclude that there exists a constant $C > 0$ such that

$$\frac{t}{\varphi(t)} \le \int_0^t \frac{ds}{\varphi(s)} \le C\frac{t}{\varphi(t)}, \quad 0 < t \le 1. \tag{2}$$

Thus, again by Krein et al. [11, Theorems 2.4.3 and 2.4.9], $\|\widetilde{\varphi}'\|_{p,\infty} \asymp \|1/\varphi\|_{p,\infty}$ and $\|\widetilde{\varphi}'\|_p \asymp \|1/\varphi\|_p$ $(p \ge 1)$. In particular, from the first of these relations and from Theorem 2.1 it follows that

$$\varphi(t) \asymp \sup_{p\ge 1} \frac{t^{1/p}}{\|1/\varphi\|_{p,\infty}}, \quad 0 < t \le 1. \tag{3}$$

(i) ⇒ (ii). If $M(\varphi) \in \mathcal{E}$, it follows then from [8, Theorem 1] that $M(\varphi) = \mathcal{L}_{L_\infty(w_1)}$. Combining this with the equality $M(\varphi) = \mathcal{U}_{L_\infty(w_2)}$, we obtain that $\mathcal{L}_{L_\infty(w_1)} = \mathcal{U}_{L_\infty(w_2)}$.

Since the condition (iii) means just equivalence of the fundamental functions of the spaces $\mathcal{L}_{L_\infty(w_1)}$ and $\mathcal{U}_{L_\infty(w_2)}$, implication (ii) ⇒ (iii) is obvious.

(iii) ⇒ (iv). We observe that by the Hölder inequality, the function M_1 given by $M_1(s) = \log \|1/\varphi\|_{1/s}$ for all $0 < s \le 1$ is convex on $(0, 1)$. Clearly,

$$\|\widetilde{\varphi}'\|_{p,\infty} = \sup_{0<t\le 1} \frac{t^{1/p}}{\varphi(t)} = \exp\left\{\sup_{0<t\le 1} \left(\frac{1}{p}\log t - \log\varphi(t)\right)\right\}, \quad 1 \le p < \infty,$$

and so the function $M_2(s) := \log \|\widetilde{\varphi}'\|_{1/s,\infty}$ for all $0 < s \le 1$ is also convex on $(0, 1)$.

In view of inequalities (2) $\|\widetilde{\varphi}'\|_{p,\infty} \asymp \|1/\varphi\|_{p,\infty}$ $(1 \le p < \infty)$, our hypothesis yields

$$\sup_{0<s\le 1} t^s e^{-M_1(s)} \asymp \sup_{0<s\le 1} t^s e^{-M_2(s)}, \quad 0 < t \le 1.$$

Let us extend the functions M_1 and M_2 to the whole real line $\mathbb{R}$ by setting $M_i(s) := +\infty$ if $s \le 0$ and $M_i(s) = M_i(1)$ if $s > 1$ $(i = 1, 2)$. Then M_1 and M_2 are decreasing convex functions on $\mathbb{R}$ and the previous equivalence can be rewritten as follows:

$$\exp\left\{\sup_{s\in\mathbb{R}}(s\log t - M_1(s))\right\} \asymp \exp\left\{\sup_{s\in\mathbb{R}}(s\log t - M_2(s))\right\}, \quad 0 < t < 1$$

and so

$$\exp\left\{M_1^*(\log t)\right\} \asymp \exp\left\{M_2^*(\log t)\right\}, \quad 0 < t < 1,$$

where M_1^* and M_2^* are the Legendre conjugate functions for M_1 and M_2, respectively. Since $M_1^*(u) = M_2^*(u) = +\infty$ if $u > 0$, the last equivalence implies that

$$M_1^*(u) - C \le M_2^*(u) \le M_1^*(u) + C, \quad u \in \mathbb{R}$$

with some positive constant C. On the other hand, by the well-known Fenchel–Moreau theorem [14, Remark 1.63, p. 26], we obtain for $i = 1, 2$

$$M_i(v) = \sup_{u\in\mathbb{R}}\left\{uv - M_i^*(u)\right\}, \quad v \in \mathbb{R},$$

and whence

$$M_1(v) - C \le M_2(v) \le M_1(v) + C, \quad v \in \mathbb{R},$$

or equivalently $\exp M_1 \asymp \exp M_2$. This yields the required equivalence

$$\|1/\varphi\|_{p,\infty} \asymp \|\widetilde{\varphi}'\|_{p,\infty} \asymp \|1/\varphi\|_p, \quad 1 \le p < \infty.$$

(iv) $\Rightarrow$ (i). It follows from the inequality in condition (iv) and from the equivalence (3) that

$$\varphi(t) \asymp \sup_{p\ge 1}\frac{t^{1/p}}{\|1/\varphi\|_p}, \quad 0 < t < 1.$$

Combining this equivalence with [8, Theorem 1], we infer that $M(\varphi) \in \mathcal{E}$. □

Example 2.5. Let $\alpha > 0$ and $\varphi_\alpha(t) := \ln^{-\alpha}(1/t)$ $(0 < t \le 1)$ and $\varphi(0) = 0$. A simple calculation shows that the following equivalences hold:

$$\|\ln^\alpha 1/t\|_{p,\infty} \asymp \alpha^\alpha p^\alpha e^{-\alpha} \quad \text{and} \quad \|\ln^\alpha 1/t\|_p \asymp p^\alpha, p \ge 1.$$

This implies that for every $\alpha > 0$

$$\|1/\varphi_\alpha\|_p \asymp \|1/\varphi_\alpha\|_{p,\infty}, \quad p \ge 1.$$

Moreover, it is easy to check that the equivalence (3) holds. Thus, it follows by Theorem 2.4 that $M(\varphi_\alpha) \in \mathcal{E}$.

Now, we present an example showing that there exist Marcinkiewicz spaces $M(\varphi)$ such that $M(\varphi) \in \mathcal{E}'$ but $M(\varphi) \notin \mathcal{E}$.

Proposition 2.6. *Let $\gamma > 0$ and let $N: (0, \infty) \to (0, \infty)$ be an increasing function such that $N(t) = t(\log t)^\gamma$ for all $t \ge t_\gamma$, where t_γ is large enough. Assume that φ is a quasi-concave function such that*

$$\varphi(t) \asymp \exp(-N^{-1}(\log(1/t))), \quad 0 < t \le 1,$$

where N^{-1} is the inverse function of N.

(i) *If $\gamma \ge 1$, then the Marcinkiewicz space $M(\varphi)$ is an extrapolation space and in addition*

$$\|x\|_{M(\varphi)} \asymp \|x\|_{L_M} \asymp \sup_{1\le p<\infty} e^{-\frac{N^*(p)}{p}}\|x\|_p,$$

where L_M is the Orlicz space, generated by the Orlicz function $M(u) = \exp(N(\log u))$ for large $u > 0$, and $N^*(p) := \sup_{s>0}\{ps - N(s)\}$.

(ii) *If $0 < \gamma < 1$, then $M(\varphi) \in \mathcal{E}' \setminus \mathcal{E}$.*

Proof. For the proof of part (i), see [6, Example 4.14]. To prove part (ii) we first observe that

$$\|\tilde{\varphi}'\|_{p,\infty} = \sup_{0<t\le 1}\frac{t^{1/p}}{\varphi(t)}, \tag{4}$$

whence

$$\log\|\tilde{\varphi}'\|_{1/s,\infty} = \sup_{0<t\le 1}\{s\log t - \log\varphi(t)\} = \sup_{y<0}\{sy - \log\varphi(e^y)\}.$$

Since the function $\log\varphi(e^y) = -N^{-1}(-y)$ is convex, then by Fenchel–Moreau theorem [14, Remark 1.63, p. 26],

$$\log\varphi(e^y) = \sup_{0<s\le 1}\{sy - \log\|\tilde{\varphi}'\|_{1/s,\infty}\},$$

and therefore

$$\varphi(t) = \sup_{0<t\le 1}\frac{t^{1/p}}{\|\tilde{\varphi}'\|_{p,\infty}}.$$

Hence, from Theorem 2.1 it follows that $M(\varphi)\in\mathcal{E}'$ and therefore $\|\tilde{\varphi}'\|_{p,\infty}\asymp\|1/\varphi\|_{p,\infty}$ (see the beginning of the proof of Theorem 2.4).

Moreover, from (4) and from the definition of the function φ we see that

$$\begin{aligned}\|\tilde{\varphi}'\|_{p,\infty} &\asymp \sup_{0<t\le 1} e^{N^{-1}(\log(1/t))}t^{1/p}\\ &= \sup_{s>0} e^{s-N(s)/p} = \left(\sup_{s>0} e^{ps-N(s)}\right)^{1/p} = e^{N^*(p)/p}.\end{aligned}$$

Applying Theorem 2.4, we conclude that to prove that $M(\varphi)\notin\mathcal{E}$ it is enough to show that

$$\lim_{p\to\infty}\left\|\frac{1}{\varphi}\right\|_p e^{-N^*(p)/p} = \infty. \tag{5}$$

Clearly, for all p large enough $N^*(p) = f_p(s_0)$, where $f_p(s) := ps - s\log^\gamma s$ and where s_0 is such that

$$p - \log^\gamma s_0 - \gamma\log^{\gamma-1}s_0 = 0.$$

Denote $x := \log s_0$. From the equation $p = x^\gamma + \gamma x^{\gamma-1}$ it follows that $x\to+\infty$ as $p\to+\infty$,

$$\frac{x^\gamma}{p}\left(1+\frac{\gamma}{x}\right) = 1, \tag{6}$$

and $x = p^{1/\gamma} + o(p^{1/\gamma})$ if $p\to+\infty$. Further, let $s_1 := s_0(1+y)$, where $y = e^{-x+p^\varepsilon}$ and $1<\varepsilon<1/\gamma$. Note that $y\to 0$ as $p\to+\infty$ and that for all p large enough we have

$$\begin{aligned}\left\|\frac{1}{\varphi}\right\|_p &\ge \left(\int_0^{\exp(-N(s_0))} e^{pN^{-1}(\log(1/t))}\,dt\right)^{1/p} = \left(-\int_{s_0}^{+\infty} e^{ps}\,de^{-N(s)}\right)^{1/p}\\ &= \left(e^{ps_0-N(s_0)} + p\int_{s_0}^{+\infty} e^{ps-N(s)}\,ds\right)^{1/p} \ge \left(\int_{s_0}^{s_1} e^{ps-N(s)}\,ds\right)^{1/p}\\ &\ge \left(e^{ps_1-N(s_1)}(s_1-s_0)\right)^{1/p} = e^{f_p(s_1)/p}(s_0y)^{1/p}\\ &= e^{f_p(s_1)/p}s_0^{1/p}\exp(-\log s_0/p + p^{\varepsilon-1}) = e^{f_p(s_1)/p}\exp(p^{\varepsilon-1}).\end{aligned}$$

Since $\varepsilon>1$, then $\exp(p^{\varepsilon-1})\to+\infty$ as $p\to+\infty$. Therefore, for proving (5) it is sufficient to show that for some $c>0$ we have

$$e^{f_p(s_1)/p} \ge ce^{f_p(s_0)/p} = ce^{N^*(p)/p},$$

or, equivalently,

$$\frac{f_p(s_1)-f_p(s_0)}{p} \ge \log c$$

for all p large enough. We claim that

$$\lim_{p\to\infty}\frac{f_p(s_1)-f_p(s_0)}{p} = 0.$$

Since $x = \log s_0$ and $s_1 = s_0(1+y)$, where $y = e^{-x+p^\varepsilon} \to 0$ as $p \to +\infty$, then taking into account (6), we obtain that

$$\begin{aligned} p^{-1}\left(f_p(s_1) - f_p(s_0)\right) &= p^{-1}\left(ps_1 - s_1\log^\gamma s_1 - ps_0 + s_0\log^\gamma s_0\right) \\ &= s_0 y - \frac{s_0}{p}\left((1+y)(\log s_0 + \log(1+y))^\gamma - \log^\gamma s_0\right) \\ &= s_0\left(y - \frac{x^\gamma}{p}\left((1+y)\left(1+\frac{\log(1+y)}{x}\right)^\gamma - 1\right)\right) \\ &= s_0\left(y - \frac{x^\gamma}{p}\left((1+y)\left(1+\frac{\gamma y}{x}+o(y^2)\right) - 1\right)\right) \\ &= s_0\left(y\left(1-\frac{x^\gamma}{p}\left(1+\frac{\gamma}{x}\right)\right) + o(y^2)\right) = s_0 \cdot o(y^2) \\ &= e^x \cdot o(e^{-2x+2p^\varepsilon}) = e^{-x+2p^\varepsilon} \cdot o(1) \to 0 \quad \text{as } p \to +\infty. \end{aligned}$$

Thus, (5) is proved and therefore $M(\varphi) \notin \mathcal{E}$. □

3. Special type extrapolation Marcinkiewicz spaces

Results of the previous section show that the condition

$$\psi(t) = \sup_{p\ge 1} \omega(p)\, t^{1/p}, \quad t \in [0,1], \tag{7}$$

where ω is a weight function defined on $[1,\infty)$, does not guarantee that the Marcinkiewicz space $M(\psi) \in \mathcal{E}$. At the same time, if ω is a function, which is tempered at infinity (i.e., $\omega(2p) \asymp \omega(p)$, $p \ge 1$), it follows from the Jawerth–Milman extrapolation theory [2] that

$$\Delta_{1\le p<\infty}\left(\omega(p)L_p\right) := \mathcal{L}_{L_\infty(\omega)} = M(\psi). \tag{8}$$

Let $\omega(p) = \varphi(e^{-p})(1 \le p < \infty)$, where φ is an increasing function on $[0,1]$ satisfying the condition: $\varphi(et) \le C\varphi(t)$ for every $0 < t \le 1/e$. Then ω is tempered at infinity if and only if $\varphi \in \Delta^2$, which means that $\varphi(t) \le C\varphi(t^2)(0 < t \le 1)$. Moreover, in this case the functions ψ and φ are equivalent on $[0,1]$ (see [8, Lemmas 1 and 2]), whence $M(\varphi) \in \mathcal{E}$.

The following theorem shows that, in fact, formula (8) holds for *every* weight function $p \mapsto \varphi(e^{-p})$ defined on $[1,\infty)$, where φ is an increasing function on $[0,1]$ with the property $\varphi(et) \le C\varphi(t)$ for every $0 < t \le 1/e$. We prove a similar assertion even for weights ω of the form $\omega(p) = \varphi(e^{-e^p})$, $1 \le p < \infty$.

Theorem 3.1. *Let $\omega(p) = \varphi(e^{-e^p})(1 \le p < \infty)$ for some increasing function φ on $[0,1]$ with the property $\varphi(et) \le C\varphi(t)(0 < t \le 1/e)$ and let the function ψ be defined by (7). Then extrapolation formula (8) holds. Moreover, the Marcinkiewicz space $M(\psi)$ coincides with the Orlicz space L_M generated by the function M defined by $M(u) = \sup_{p\ge1} \omega(p)^p u^p$ for all $u \ge 0$.*

Proof. We first note that for $p \in [\log k, \log(k+1)]$, $k \ge 2$, we have

$$\varphi(e^{-e^p})\,\|x\|_p \le \varphi(e^{-k})\,\|x\|_{\log(k+1)} \le C\varphi(e^{-(k+1)})\,\|x\|_{\log(k+1)},$$

which yields

$$\sup_{p>1} \varphi(e^{-e^p})\,\|x\|_p \asymp \sup_{k\ge3,\,k\in\mathbb{N}} \varphi(e^{-k})\,\|x\|_{\log k}. \tag{9}$$

Further, it is clear that $\mathcal{L}_{L_\infty(\omega)}$ is an r.i. space on I with the fundamental function ψ. Since $M(\psi)$ is the largest space among all r.i. spaces with a given fundamental function [11, Theorem 2.5.7], we get the continuous embedding

$$\mathcal{L}_{L_\infty(\omega)} \hookrightarrow M(\psi).$$

To complete the proof of (8) it is enough to show that the reverse continuous embedding holds. Since

$$x^*(t) \le \frac{1}{\psi(t)}\|x\|_{M(\psi)}, \quad t \in (0,1],$$

for every $x \in M(\psi)$, the latter embedding is an immediate consequence of the fact that $1/\psi \in \mathcal{L}_{L_\infty(\omega)}$, or, equivalently, by (9),

$$\sup_{k\ge3} \varphi(e^{-k}) \left\|\frac{1}{\psi}\right\|_{\log k} < \infty.$$

From the definition of ψ it follows that

$$\psi(t) \ge \varphi(e^{-(k+1)}) t^{\frac{1}{\log(k+1)}} \ge \frac{1}{C}\varphi(e^{-k}) t^{\frac{1}{\log(k+1)}},$$

and so

$$\frac{1}{\psi(t)} \le \frac{C}{\varphi(e^{-k})} t^{-\frac{1}{\log(k+1)}}.$$

Combining these estimates gives

$$\varphi(e^{-k}) \left\| \frac{1}{\psi} \right\|_{\log k} \le C \left(\int_0^1 t^{-\frac{\log k}{\log(k+1)}} dt \right)^{\frac{1}{\log k}} = C \left(\frac{\log(k+1)}{\log(1+1/k)} \right)^{\frac{1}{\log k}} < C_1 < \infty,$$

and (8) is proved.

We now prove that the space $\mathcal{L}_{L_\infty(\omega)}$ coincides with the Orlicz space L_M. We first observe that $\int_0^1 M(|x(t)|)\, dt \le 1$, provided $\|x\|_{L_M} \le 1$. Then, using the definition of M, we deduce that

$$\omega(p)^p \int_0^1 |x(t)|^p\, dt \le 1, \quad 1 < p < \infty.$$

Hence, it follows that $x \in \mathcal{L}_{L_\infty(\omega)}$ and $\|x\|_{\mathcal{L}_{L_\infty(\omega)}} \le 1$.

Conversely, let $\|x\|_{\mathcal{L}_{L_\infty(\omega)}} \le 1$. Without loss of generality, we may assume that $\varphi(1) = 1$. By this assumption, $C^{-k} \le \varphi(e^{-k}) \le 1$ and $\varphi(e^{-k})^{\log k} \le C\varphi(e^{-k})^{\log(k+1)}$. This implies

$$\begin{aligned} M(u) &= \sup_{1\le p<\infty} \varphi(e^{-e^p})^p u^p \le \sup_{k\ge 3} C^{1+\log k} \varphi(e^{-k})^{\log k} u^{\log k} \\ &\le \sum_{k=3}^{\infty} C^{2\log k} \varphi(e^{-k})^{\log k} u^{\log k}, \end{aligned}$$

and whence

$$\int_0^1 M\left(\frac{|x(t)|}{e^2C^2}\right) dt \le \sum_{k=3}^{\infty} C^{2\log k}\varphi(e^{-k})^{\log k} \int_0^1 \left(\frac{|x(t)|}{e^2C^2}\right)^{\log k} dt \le \sum_{k=3}^{\infty} \frac{1}{k^2} < 1.$$

This shows that $\|x\|_{L_M} \le e^2C^2$. $\square$

The following statement is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let $\omega(p) = \varphi(e^{-p})$ $(1 \le p < \infty)$ for some increasing function φ on $[0, 1]$ with the property $\varphi(et) \le C\varphi(t)$ $(0 < t \le 1/e)$ and let the function ψ be defined by (7). Then extrapolation formula (8) holds. Moreover, $M(\psi)$ coincides with the Orlicz space L_M generated by the function $M(u) = \sup_{p\ge1} \omega(p)^p u^p$ $(u \ge 0)$.*

Let $\omega(p) = \varphi(e^{-p})$ $(1 \le p < \infty)$, where φ is a function satisfying the conditions from Corollary 3.2. An inspection of the proof of Theorem 3.1 shows that relation (8) holds if and only if there is a constant $C > 0$ such that

$$\|1/\psi\|_p \le \frac{C}{\varphi(e^{-p})}, \quad 1 \le p < \infty. \tag{10}$$

From the properties of φ it follows that the right-hand side of (10) does not exceed C_1^p. Let us present an example of an extrapolation Marcinkiewicz space $M(\psi)$ whose fundamental function ψ does not satisfy any inequality of the form (10) and which therefore cannot be representable as in (7) with $\omega(p) = \varphi(e^{-p})$, where φ satisfies the conditions from Corollary 3.2.

Example 3.3. Let $0 < \theta < 1$ and $\rho_\theta(t) := \exp\left(-\ln^\theta(1/t)\right)$ if $0 < t \le 1$ and $\rho_\theta(0) = 0$. Since $\lim_{t\to0+} \rho_\theta(t) = 0$ and for sufficiently small $t > 0$ $\rho_\theta'(t) > 0$ and $\rho_\theta''(t) < 0$, there exists an increasing concave function $\psi_\theta(t)$ on $[0, 1]$, $\psi(0) = 0$, which is equivalent to $\rho_\theta(t)$. From [8, Example 1] it follows that $M(\psi_\theta) \in \mathcal{E}$ and

$$\|1/\psi_\theta\|_p \asymp \exp\left(C_\theta p^{\theta/(1-\theta)}\right).$$

This shows that in the case $\theta \in (1/2, 1)$ inequality (10) cannot be fulfilled and therefore the fundamental function $\psi_\theta(t)$ of the corresponding Marcinkiewicz space cannot be representable in the form $\sup_{p\ge1} \varphi(e^{-p})\, t^{1/p}$ $(0 < t \le 1)$, with some increasing function φ on $[0, 1]$ with the property $\varphi(et) \le C\varphi(t)$ $(0 < t \le 1/e)$. On the other hand, it is easy to check that from Theorem 3.1 it follows that

$$\|x\|_{M(\psi_\theta)} \asymp \sup_{p\ge1} \frac{\|x\|_p}{\exp\left(C_\theta p^{\theta/(1-\theta)}\right)}.$$

It is interesting to note that any extrapolation functor of the type Δ generated by the discrete scale $\{L_k\}_{k=1}^{\infty}$ of L_p-spaces gives an Orlicz space [6, section 5.1]. More precisely, for an arbitrary positive sequence $\{\omega(k)\}_{k=1}^{\infty}$, we have that $L_\Phi = \Delta^d(\omega L_k)$, where

$$\|x\|_{\Delta^d(\omega L_k)} := \sup_{k\in\mathbb{N}} \omega(k)\|x\|_k \quad \text{and} \quad \Phi(u) = \sup_{k\in\mathbb{N}} \omega(k)^k u^k. \tag{11}$$

In contrast to that the latter space may not coincide with any Marcinkiewicz space.

Proposition 3.4. *Let $w(k) = e^{-e^{e^k}}$ for each $k \in \mathbb{N}$. Then the space $\Delta^d(\omega L_k)$ defined in (11) does not coincide with any Marcinkiewicz space.*

Proof. Since $\Delta^d(\omega L_k) = L_\Phi$ with the function Φ defined in (11), we need only to show that $L_\Phi \neq M(\varphi)$, where $\varphi(t) := \sup_k w(k)t^{1/k} \asymp \varphi_{L_\Phi}(t)$. In view of [15] (see also [16]), it is enough to check that $1/\varphi \notin L_\Phi$, i.e., for any $\lambda > 1$

$$\int_0^1 \Phi\left(\frac{1}{\lambda\varphi(t)}\right) dt = \infty. \tag{12}$$

Let the function $f(s) = e^{e^s} + \frac{1}{s}\log(1/t)$ take its maximum on the semi-axis $(0, \infty)$ at the point s_0. Then

$$s_0^2 e^{s_0} e^{e^{s_0}} = \log(1/t),$$

whence $s_0 < \log\log\log(1/t)$. Therefore,

$$\varphi(t) \le Cw(k_1)t^{1/k_1} \quad \text{with } k_1 \in \mathbb{N} \text{ such that } k_1 < \log\log\log(1/t) + 1.$$

By (11), we have

$$\Phi\left(\frac{1}{\lambda\varphi(t)}\right) \ge w(k_1)^{k_1} \frac{1}{C^{k_1}\lambda^{k_1}(w(k_1)t^{1/k_1})^{k_1}} = \frac{1}{C^{k_1}\lambda^{k_1}t}.$$

Let $C\lambda = e^a$, where $a > 0$. Since $k_1 < \log\log\log(1/t) + 1$, then

$$\frac{1}{C^{k_1}\lambda^{k_1}} > \frac{1}{(C\lambda)^{\log\log\log(1/t)+1}} = \frac{1}{C\lambda\log^a\log(1/t)}.$$

Thus,

$$\int_0^1 \Phi\left(\frac{1}{\lambda\varphi(t)}\right) dt \ge \frac{1}{C\lambda}\int_0^1 \frac{1}{t\log^a\log(1/t)}\, dt = \infty,$$

and the relation (12) is proved. $\square$

4. Strong extrapolation Orlicz–Lorentz spaces

In [9], an interesting subclass of $\mathcal{E}$ has been introduced and studied. If X is an r.i. space on I, then by $\widetilde{X}$ we denote the Banach lattice of all measurable functions f on $[1, \infty)$ such that

$$\|f\|_{\widetilde{X}} := \|f(\log_2(2/t))\|_X < \infty.$$

An r.i. space X is called a *strong extrapolation space* ($X \in \mathcal{S}\mathcal{E}$) if $X = \mathcal{L}_{\widetilde{X}}$ (with equivalence of norms).

One of the main results of the article [9] is a characterization of strong extrapolation spaces in terms of boundedness of the composition operator $Sx(t) = x(t^2)$.

Recall (see Introduction) that, for arbitrary Orlicz function M on $[0, \infty)$ and concave function φ on I, the Orlicz–Lorentz space $\Lambda_M(\varphi)$ consists of all $x \in L^0(I)$ for which there exists $\lambda > 0$ such that

$$\int_0^1 M\left(x^*(t)/\lambda\right) d\varphi(t) < \infty.$$

$\Lambda_M(\varphi)$ is an r.i. space equipped with the norm

$$\|x\| := \inf\left\{\lambda > 0\colon \int_0^1 M\left(x^*(t)/\lambda\right) d\varphi(t) \le 1\right\}.$$

Proposition 4.1. *Let $E := \Lambda_M(\varphi)$ be an Orlicz–Lorentz space on I with $\varphi(0+) = 0$, and let F be the Orlicz space L_M on $([1, \infty), \mu)$ with $\mu = w\,dm$, where $w(s) := \varphi'(2^{-s})2^{-s}$ for all $s \in [1, \infty)$. Then we have the following embeddings*

$$\mathcal{L}_F \hookrightarrow E, \qquad \mathcal{L}_{\widetilde{E}} \hookrightarrow \mathcal{L}_F.$$

Proof. To prove the first embedding we fix $0 \neq x \in \mathcal{L}_F$. Then for $\lambda := \|x\|_{\mathcal{L}_F}$, we have

$$\int_1^\infty M\left(\|x\|_s/\lambda\right)\, w(s)\, ds \le 1.$$

Since $x \in L_s$ for all $s \ge 1$ and $x^*(2^{-s}) \le 2\|x\|_s$, we obtain

$$\begin{aligned}\int_0^{1/2} M\left(x^*(t)/2\lambda\right)\varphi'(t)\,dt &= \log 2 \int_1^\infty M\left(x^*(2^{-s})/2\lambda\right) 2^{-s}\varphi'(2^{-s})\,ds \\ &\le \int_1^\infty M\left(\|x\|_s/\lambda\right)\, w(s)\,ds \le 1.\end{aligned}$$

This implies that

$$\|x^*\chi_{[0,1/2]}\|_E \le 2\|x\|_{\mathcal{L}_F}.$$

Now observe that for any r.i. space E on I we have

$$\|x\|_E \le \|x^*\chi_{[0,1/2]}\|_E + \|x^*\chi_{(1/2,1]}\|_E \le 2\|x^*\chi_{[0,1/2]}\|_E.$$

Thus combining the obtained estimates we conclude that

$$\|x\|_E \le 4\|x\|_{\mathcal{L}_F},$$

which yields the first embedding.

Let us prove the second embedding. If $0 \neq x \in \mathcal{L}_{\widetilde{E}}$, then for $\lambda := \|x\|_{\mathcal{L}_{\widetilde{E}}}$, since M is an Orlicz function, we have

$$\begin{aligned}1 \ge \int_0^1 M\left((\|x\|_{\log_2 2/t})^*/\lambda\right)\varphi'(t)\,dt &\ge \int_0^{1/2} M\left((\|x\|_{\log_2 1/t})/\lambda\right)\varphi'(t)\,dt \\ &= \log 2\int_1^\infty M\left(\|x\|_s/\lambda\right)\,\varphi'(2^{-s})2^{-s}\,ds \\ &\ge \int_1^\infty M\left(\log 2\,\|x\|_s/\lambda\right)\, w(s)\,ds.\end{aligned}$$

Hence $x \in \mathcal{L}_F$ with $\|x\|_{\mathcal{L}_F} \le (\log 2)^{-1}\,\|x\|_{\mathcal{L}_{\widetilde{E}}}$, and this completes the proof. □

Further, we use the standard notions from the interpolation theory (see, e.g. [17]). A mapping $\mathcal{F}$ acting on the class of all couples of Banach lattices is called a *positive interpolation functor* if for every couple $\overline{X} = (X_0, X_1)$ of Banach lattices $\mathcal{F}(\overline{X})$ is an intermediate Banach lattice with respect to $\overline{X}$ (i.e., $X_0 \cap X_1 \hookrightarrow \mathcal{F}(\overline{X}) \hookrightarrow X_0 + X_1$), and $T: \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})$ for every linear positive operator T (i.e., $Tx \ge 0$ whenever $x \ge 0$) such that $T: \overline{X} \to \overline{Y}$ (meaning that $T: X_0 + X_1 \to Y_0 + Y_1$ and its restrictions $T: X_j \to Y_j (j = 0, 1)$ are defined and being linear positive operators). If, in addition, there is a constant $C > 0$ such that for each $T: \overline{X} \to \overline{Y}$ we have

$$\|T\|_{\mathcal{F}(\overline{X})\to\mathcal{F}(\overline{Y})} \le C\max\{\|T\|_{X_0\to Y_0}, \|T\|_{X_1\to Y_1}\},$$

then $\mathcal{F}$ is called *bounded* (an *exact* if $C = 1$).

We also need the definition of the Calderón–Lozanovskii spaces.

Recall that if $\overline{X} = (X_0, X_1)$ is a Banach couple of lattices on (Ω, μ) and $\phi: [0,\infty) \times [0,\infty) \to [0,\infty)$ is a concave and positively homogeneous function (that is, $\phi(\lambda s, \lambda t) = \lambda\phi(s,t)$ for all $\lambda, s, t \ge 0$) with $\phi(0,0) = 0$ (for short, $\phi \in \mathcal{P}$), then the Calderón–Lozanovskii space $\phi(\overline{X}) := \phi(X_0, X_1)$ consists of all $x \in L^0(\mu)$ such that $|x| \le \lambda\phi\,(|x_0|, |x_1|)\mu$-a.e. on Ω for some $x_j \in X_j$ with $\|x_j\|_{X_j} \le 1, j = 0, 1$. The space $\phi(\overline{X})$ is a Banach lattice equipped with the norm (see [18,19])

$$\|x\|_{\phi(\overline{X})} := \inf\left\{\lambda > 0: |x| \le \lambda\phi(|x_0|, |x_1|),\ \|x_0\|_{X_0} \le 1,\ \|x_1\|_{X_1} \le 1\right\}.$$

In the case of the power function $\phi(s,t) = s^{1-\theta}t^\theta$ with $0 < \theta < 1$, $\phi(\overline{X})$ is the well-known Calderón space denoted by $X_0^{1-\theta}X_1^\theta$ (see [20]). Recall that $\mathcal{F}(\cdot) := \phi(\cdot)$ is an exact positive interpolation functor [21].

Let $M: [0,\infty) \to [0,\infty)$ be an Orlicz function. Put $\phi(s,t) := tM^{-1}(s/t)$ if $t > 0$ and $\phi(s,t) = 0$ if $t = 0$, where M^{-1} is the right-continuous inverse of M. Then it is easy to check that for any Lorentz space $\Lambda(\varphi)$, we have

$$\phi(\Lambda(\varphi), L_\infty) = \Lambda_M(\varphi). \tag{13}$$

Theorem 4.2. *Let $E := \Lambda_M(\varphi)$ be an Orlicz–Lorentz space on I with $\varphi(0+) = 0$. Then the following statements are true.*

(i) *If the composition operator $Sx(t) = x(t^2)$ is bounded in E, then E is a strong extrapolation space and $E = \mathcal{L}_{\widetilde{E}} = \mathcal{L}_F$, where F is as in Proposition 4.1.*

(ii) *Assume φ satisfies Δ^2-condition ($\varphi \in \Delta^2$ for short), i.e., there exists a constant $C > 0$ such that $\varphi(t) \le C\varphi(t^2)$ for all $t \in I$. Then E is a strong extrapolation space.*

(iii) *If E is separable, then E is a strong extrapolation space if and only if $\varphi \in \Delta^2$.*

Proof. (i). From [9, Theorem 2.3], our hypothesis implies that E is a strong extrapolation space, i.e., $E = \mathcal{L}_{\widetilde{E}}$. Thus the required statement follows from Proposition 4.1.

(ii). If $\varphi \in \Delta^2$, then the composition operator S is bounded in $\Lambda(\varphi)$ (see [9, Proposition 2.9]). Since there exists $\phi \in \mathcal{P}$ such that we have (13) and the map $\mathcal{F}(\cdot) := \phi(\cdot)$ is a positive interpolation functor, we conclude that the composition operator S is bounded in E. It then follows from [9, Theorem 2.3] that $E \in \mathcal{S}\mathcal{E}$.

(iii). The separability of E implies that there exist a constant $C > 0$ and $t_0 > 0$ such that $M(2t) \le CM(t)$ for all $t > t_0$ (see [22]) or, equivalently, $\gamma_\psi > 0$, where $\psi(t) := 1/M^{-1}(1/t)$ for all $t \in I$ is the fundamental function of the Orlicz space L_M.

Assume that E is a strong extrapolation space. Then it follows from [9, Theorem 2.3 and Proposition 2.9] that $\varphi_E = \psi(\varphi) \in \Delta^2$. Combining this with the separability of E easily yields that $\varphi \in \Delta^2$. In fact, otherwise, for any $h > 0$ there is a $t \in (0, 1)$ for which $\varphi(t^2) \le h\varphi(t)$. Then, by the fact that $\gamma_\psi > 0$, there are $C > 0$ and $\varepsilon > 0$ such that

$$\frac{\psi(\varphi(t^2))}{\psi(\varphi(t))} \le \frac{\psi(h\varphi(t))}{\psi(\varphi(t))} \le Ch^\varepsilon.$$

Since $h > 0$ is arbitrary, this contradicts $\varphi_E \in \Delta^2$ and (iii) is proved. □

Remark 4.3. In particular, if $1 \le r < \infty$ and $M(t) = t^r$ for all $t \ge 0$, we obtain the space $\Lambda_r(\varphi)$ equipped with the norm

$$\|x\|_{\Lambda_r(\varphi)} := \left(\int_0^1 \left(x^*(t)\right)^r \, d\varphi(t)\right)^{1/r}.$$

From Theorem 4.2 it follows that

$$\|x\|_{\Lambda_r(\varphi)} \asymp \left(\sum_{k=1}^\infty \|x\|_k^r \, \varphi'(2^{-k})\, 2^{-k}\right)^{1/r} \tag{14}$$

whenever $\varphi \in \Delta^2$. In a special case $\varphi(t) = \log^{1-r}(e/t)(r > 1)$ we obtain

$$\|x\|_{\Lambda_r(\log^{1-r}(e/t))} \asymp \left(\sum_{k=1}^\infty \left(\frac{\|x\|_k}{k}\right)^r\right)^{1/r}.$$

We note that the above formula was proved earlier in [23], where one can also find its applications in studying convolution inequalities corresponding to the critical case of Sobolev embedding. If φ is a concave increasing function satisfying the condition $\varphi'(t) \le Ct\varphi'(t^2)$ (which is stronger than the condition $\varphi \in \Delta^2$) formula (14) was obtained in [24]. If $\varphi \in \Delta^2$ is arbitrary, then (14) was proved in [8].

The problem of describing strong extrapolation non-separable Orlicz–Lorentz spaces is still open. We note that if $\psi(t) := 1/M^{-1}(1/t) \in \Delta^2$, then the Orlicz space L_M is a strong extrapolation and coincides with the Marcinkiewicz space $M(\psi)$ [6, Section 4]. Therefore, if φ_E is the fundamental function of the Orlicz–Lorentz space $E = \Lambda_M(\varphi)$, then for some $\lambda > 0$

$$\begin{aligned}\int_0^1 M\left(\frac{1}{\lambda\varphi_E(t)}\right) d\varphi(t) &= \int_0^1 M\left(\frac{1}{\lambda\psi(\varphi(t))}\right) d\varphi(t)\\ &= \int_0^1 M\left(\frac{1}{\lambda\psi(s)}\right) ds < \infty,\end{aligned}$$

which implies that $1/\varphi_E \in \Lambda_M(\varphi)$. Hence, the space $\Lambda_M(\varphi)$ coincides with the Marcinkiewicz space $M(\varphi_E)$. However, as is shown in the following proposition, this space may not be a strong extrapolation.

Proposition 4.4. *For an arbitrary increasing concave continuous function ψ on I with $\psi(0) = 0$ and $\psi(1) = 1$ there is a function φ with the same properties such that the function $\psi(\varphi) \notin \Delta^2$.*

Proof. For a given ψ we choose inductively the points $1 = t_0 > t_1 > t_1^2 > t_2 > t_2^2 > t_3 > \cdots > 0$ and the numbers $1 = b_0 > b_1 > b_2 > \cdots > 0$ and $0 = k_0 < k_1 < k_2 < \cdots$ with $\lim_{n\to\infty} b_n = 0$ and $k_m t_{m+1} + b_m = k_{m+1}t_{m+1} + b_{m+1}$ so that the function $\psi(\varphi)$, where the piecewise linear function φ defined by $\varphi(t) = k_m t + b_m$ if $t \in [t_{m+1}, t_m]$ $(m = 0, 1, 2, \ldots)$ and $\varphi(0) = 0$, possesses required properties.

First, we set $\varphi(t_0) = 1$. Now, suppose that $1 = t_0 > t_1 > t_1^2 > t_2 > t_2^2 > t_3 > \cdots > t_{n-1}^2 > t_n$, $1 = b_0 > b_1 > b_2 > \cdots > b_n$, and $0 = k_0 < k_1 < k_2 < \cdots < k_n$ are already chosen. Let $n \geq 0$ and $t_{n+1} \in (0, t_n^2)$ satisfies the conditions

$$k_n t_{n+1}^2 + b_n t_{n+1} < \frac{b_n}{4} \tag{15}$$

and

$$\psi\left(2(k_n t_{n+1}^2 + b_n t_{n+1})\right) < \frac{b_n}{n+1} \tag{16}$$

and k_{n+1} and b_{n+1} be such that the points with coordinates $(t_{n+1}, k_n t_{n+1} + b_n)$ and $(t_{n+1}^2, 2(k_n t_{n+1}^2 + b_n t_{n+1}))$ lie on the straight line $y = k_{n+1} t + b_{n+1}$. Then an easy calculation shows that $k_{n+1} > k_n (n = 0, 1, 2, \ldots)$, $b_{n+1} > 0$ and by (15) $b_{n+1} \leq b_n/2 (n = 0, 1, 2, \ldots)$, whence $b_n \downarrow 0$. Moreover, $\psi(\varphi(t_{n+1})) = \psi(k_n t_{n+1} + b_n) \geq k_n t_{n+1} + b_n > b_n$ and from (16) it follows that $\psi(\varphi(t_{n+1}^2)) < b_n/(n+1)$. Therefore, $\psi(\varphi) \notin \Delta^2$, and the proof is complete. $\square$

It should be noted that the condition $\varphi_E \in \Delta^2$, where $E = \Lambda_M(\varphi)$, can be fulfilled when neither of the conditions $\varphi \in \Delta^2$ and $\psi(t) := 1/M^{-1}(1/t) \in \Delta^2$ holds.

Example 4.5. Let φ be an increasing concave function on I such that $\varphi(t) \asymp \exp(-\log^2 \log 1/t)(0 < t \leq 1/e)$. It is not hard to check that $\varphi(\varphi) \in \Delta^2$ but $\varphi \notin \Delta^2$.

5. On interpolation of extrapolation spaces

We will need a special variant of a result from [25]. For the sake of completeness we include its proof. We use the most general version of what is known as the *Hahn–Banach extension theorem* [26, Theorem 2.1].

If $(X, \leq)$ is a vector lattice, then a linear operator $S: X \to L^0(\mu)$ is called *positive* if $S(x) \geq 0$ whenever $x \in X$ and $x \geq 0$.

Theorem 5.1. *Let (X_0, X_1) and (Y_0, Y_1) be couples of Banach lattices and let $T: X_0 + X_1 \to Y_0 + Y_1$ be a sublinear operator such that $|Tx| \leq T|x|$ $(x \in X_0 + X_1)$. Suppose that $\|Tx\|_{Y_j} \leq C_j \|x\|_{X_j}$ for any $x \in X_j$ $(j = 0, 1)$. Then, for every exact positive interpolation functor $\mathcal{F}$, T maps $\mathcal{F}(X_0, X_1)$ into $\mathcal{F}(Y_0, Y_1)$ and*

$$\|Tx\|_{\mathcal{F}(Y_0,Y_1)} \leq \max\{C_0, C_1\} \|x\|_{\mathcal{F}(X_0,X_1)}, \quad x \in \mathcal{F}(X_0, X_1).$$

Proof. Let $p: X_0 + X_1 \to L^0(\mu)$ be defined by $p(x) := T(x^+)$ for every $x \in X_0 + X_1$. Then we have that $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in (X_0 + X_1)$, $\lambda \geq 0$. Fix $x_0 \in \mathcal{F}(X_0, X_1)$ and put $H := \operatorname{span}\{|x_0|\}$. Define the positive linear operator $S_0: H \to L^0(\mu)$ by

$$S_0(\lambda |x_0|) = \lambda T(|x_0|), \quad \lambda \in \mathbb{R}.$$

Clearly, we have $S_0(x) \leq p(x)$ for all $x \in H$. Now, by Hahn–Banach extension Theorem [26, Theorem 2.1], there exists a linear extension S of S_0 to the whole space $X_0 + X_1$ satisfying $S(x) \leq p(x)$ for all $x \in X_0 + X_1$. In particular, if $0 \leq x \in X_0 + X_1$, then

$$-S(x) = S(-x) \leq p(-x) = T((-x)^+) = T(0) = 0$$

implies that $S(x) \geq 0$. In consequence, S is a positive extension S_0 to the whole $X_0 + X_1$. Since $S(x) \leq p(x) = T(x^+)$ and $|S(x)| \leq S(|x|)$ for $x \in X_0 + X_1$ (we note that S is a linear positive operator), our hypothesis on T yields

$$\|Sx\|_{Y_j} \leq \|T(|x|)\|_{Y_j} \leq C_j \|x\|_{X_j}$$

for all $x \in X_j$ $(j = 0, 1)$. Combining these remarks with interpolation property, we obtain $T(|x_0|) = S(|x_0|) \in \mathcal{F}(Y_0, Y_1)$ and therefore

$$\|T(|x_0|)\|_{\mathcal{F}(Y_0,Y_1)} = \|S(|x_0|)\|_{\mathcal{F}(Y_0,Y_1)} \leq \max\{C_0, C_1\} \|x_0\|_{\mathcal{F}(X_0,X_1)}.$$

Since $|T(x_0)| \leq T(|x_0|)$ we obtain the desired inequality. $\square$

We show applications of the above theorem which are independent of interest.

Corollary 5.2. *Let (E_0, E_1) be a Banach couple of lattices on (Ω, μ). Let $\mathcal{X} \subset L^0(\nu)$ be a linear subspace and let $S: \mathcal{X} \to L^0(\mu)$ be a sublinear operator vanishing only at zero such that $|Sx| \leq S|x|$ $(x \in \mathcal{X})$. Assume that $(D_{E_0}(S), D_{E_1}(S))$ be a Banach couple of lattices. Then, for every exact positive interpolation functor $\mathcal{F}$,*

$$\mathcal{F}(D_{E_0}(S), D_{E_1}(S)) \hookrightarrow D_{\mathcal{F}(E_0,E_1)}(S).$$

Proof. Since $\|Sx\|_{E_j} = \|x\|_{D_{E_j}(S)}$ whenever $x \in D_{E_j}(S)(j = 0, 1)$, the operator $S: D_{E_0}(S) + D_{E_1}(S) \to E_0 + E_1$ satisfies all conditions of Theorem 5.1. Hence $S: \mathcal{F}(D_{E_0}(S), D_{E_1}(S)) \to \mathcal{F}(E_0, E_1)$ with

$$\|Sx\|_{\mathcal{F}(E_0,E_1)} \le \|x\|_{\mathcal{F}(D_{E_0}(S),D_{E_1}(S))}$$

for all $x \in \mathcal{F}(D_{E_0}(S), D_{E_1}(S))$, which yields the desired embedding. $\square$

Corollary 5.3. *Let $\mathcal{F}$ be an exact positive interpolation functor. Then, we have the following.*

(i) *If (F_0, F_1) is a Banach couple of lattices on $([1, \infty), m)$, then*

$$\mathcal{F}(\mathcal{L}_{F_0}, \mathcal{L}_{F_1}) \hookrightarrow \mathcal{L}_{\mathcal{F}(F_0,F_1)} \quad \text{and} \quad \mathcal{F}(\mathcal{U}_{F_0}, \mathcal{U}_{F_1}) \hookrightarrow \mathcal{U}_{\mathcal{F}(F_0,F_1)}.$$

(ii) *If (E_0, E_1) is a couple of r.i. spaces on I, then*

$$\mathcal{F}(\mathcal{L}_{\widetilde{E_0}}, \mathcal{L}_{\widetilde{E_1}}) \hookrightarrow \mathcal{L}_{\mathcal{F}(\widetilde{E_0},\widetilde{E_1})} = \mathcal{L}_{\widetilde{\mathcal{F}(E_0,E_1)}}.$$

Corollary 5.4. *Let a positive function $w \in L_1([1, \infty), m)$ and let φ be a concave function on $[0, 1]$ defined by*

$$\varphi(t) = \int_1^\infty t^{1/p}\, w(p)\, dp, \quad 0 \le t \le 1.$$

Then, for any Orlicz function M on $[0, \infty)$ the following continuous inclusion holds:

$$\Lambda_M(\varphi) \hookrightarrow \mathcal{L}_F,$$

where F is the Orlicz space L_M on $([1, \infty), \mu)$ with $d\mu = w\, dm$.

Proof. Let $1 \le p < \infty$ and L_p be defined on $[0, 1]$. Since $\varphi_{L^p}(t) = t^{1/p}$ for $0 \le t \le 1$, it follows from [11, Theorem 5.5] that $L_{p,1} := \Lambda(\varphi_{L_p}) \hookrightarrow L_p$ with

$$\|x\|_p \le \|x\|_{p,1} := \int_0^1 x^*(t)\, dt^{1/p}, \quad x \in L_{p,1}.$$

For every x from the Lorentz space $\Lambda(\varphi)$ on $[0, 1]$, we have

$$\begin{aligned}\int_1^\infty \|x\|_p\, w(p)\, dp &\le \int_1^\infty \left(\int_0^1 x^*(t)\, dt^{1/p}\right) w(p)\, dp \\ &= \int_0^1 x^*(t) \int_1^\infty \frac{t^{1/p-1}}{p}\, w(p)\, dp\, dt = \int_0^1 x^*(t)\, d\varphi(t).\end{aligned}$$

Clearly,

$$\sup_{1\le p<\infty} \|x\|_p = \sup_{0<t\le 1} x^*(t) = \|x\|_\infty, \quad x \in L_\infty.$$

In consequence, we obtain

$$\Lambda(\varphi) \hookrightarrow \mathcal{L}_{F_0}, \qquad L_\infty \hookrightarrow \mathcal{L}_{F_1},$$

where $F_0 = L_1(w)$, $F_1 = L_\infty$ are Banach latices on $([1, \infty), m)$.

Now put $\phi(s, t) := tM^{-1}(s/t)$ if $t > 0$ and $\phi(s, t) = 0$ if $t = 0$, where M^{-1} is the right-continuous inverse of M. Then the following formulas hold with equalities of norms:

$$\phi(\Lambda(\varphi), L_\infty) = \Lambda_M(\varphi), \qquad \phi(F_0, F_1) = L_M(\mu).$$

Combining the above relations with Corollary 5.3 applied to the positive interpolation functor $\mathcal{F}(\cdot) := \phi(\cdot)$ yields

$$\Lambda_M(\varphi) \hookrightarrow \phi\left(\mathcal{L}_{F_0}, \mathcal{L}_{F_1}\right) \hookrightarrow \mathcal{L}_{\phi(F_0,F_1)} = \mathcal{L}_F,$$

where $F = L_M(\mu)$ and this completes the proof. $\square$

We conclude the paper with the following remark that if $\varphi \in \Delta^2$, then it is easy to check that for w defined by $w(p) = 2^{-p}\varphi'(2^{-p})$ for all $1 \le p < \infty$,

$$\varphi(t) \asymp \int_1^\infty t^{1/p}\, w(p)\, dp.$$

Thus, applying Proposition 4.1 and Theorem 4.2, we deduce that $\Lambda_M(\varphi) = \mathcal{L}_F$.

Acknowledgments

Astashkin and Lykov were partially supported by RFBR grant no. 10-01-00077. Mastyło was partially supported by the National Science Centre (NCN), Poland, grant no. 2011/01/B/ST1/06243.

References

[1] S. Yano, An extrapolation theorem, J. Math. Soc. Japan. 3 (1) (1951) 296–305.
[2] B. Jawerth, M. Milman, Extrapolation theory with applications, Mem. Amer. Math. Soc. 89 (440) (1991) 82.
[3] M. Milman, Extrapolation and Optimal Decompositions with Applications to Analysis, in: Lecture Notes in Math., vol. 1580, Springer-Verlag, Berlin, 1994, p. 162.
[4] M.J. Carro, From restricted weak type to strong type estimates, J. London Math. Soc. (2) 70 (3) (2004) 750–762.
[5] M.J. Carro, L. Colzani, G. Sinnamon, From restricted type to strong type estimates on quasi-Banach rearrangement invariant spaces, Studia Math. (1) 182 (1) (2007) 1–27.
[6] S.V. Astashkin, K.V. Lykov, Extrapolation description of rearrangement invariant spaces and related problems, Proceedings of International Symposium on Banach and Function Spaces III (Kitakyushu, Japan, 2009), Yokohama Publisher (2011), pp. 1–52 (in press).
[7] G.E. Karadzhov, M. Milman, Extrapolation theory: new results and applications, J. Approx. Theory 133 (2005) 38–99.
[8] S.V. Astashkin, K.V. Lykov, Extrapolation description of Lorentz and Marcinkiewicz spaces close to L_∞, Sibirsk, Mat. Zh. 47 (5) (2006) 974–992. (Russian); English translation in; Sib. Math. J. 47 (5) (2006) 797–812.
[9] S.V. Astashkin, K.V. Lykov, Strong extrapolation spaces and interpolation, Sibirsk. Mat. Zh. 50 (2) (2009) 250–266. English translation in; Sib. Math. J. 50 (2) (2009) 199–213.
[10] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press, Boston, 1988.
[11] S.G. Krein, Ju.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, in: Translations of Mathematical Monographs, vol. 54, American Mathematical Society, Providence R.I, 1982.
[12] M. Mastyło, Banach spaces via sublinear operators, Math. Japon. 36 (1) (1991) 85–92.
[13] K.V. Lykov, Extrapolation in a scale of L_p-spaces and the convergence of orthogonal series in Marcinkiewicz spaces, Vestn. Samar. Gos. Univ. Estestvennonauchn. Ser. (2) (2006) 28–43. (Russian).
[14] S. Simons, Minimax and Monotonicity, in: Lecture Notes in Math., vol. 1693, Springer-Verlag, Berlin, 1998, p. 172.
[15] G.G. Lorentz, Relations between function spaces, Proc. Amer. Math. Soc. 12 (1961) 127–132.
[16] Ya.B. Rutickii, On some classes of measurable functions, Uspekhi Matemat. Nauk 20 (4) (1965) 205–208. (Russian).
[17] Yu.A. Brudnyi, N.Ya. Krugljak, Interpolation Functors and Interpolation Spaces *I*, North-Holland, Amsterdam, 1991.
[18] G.Ya. Lozanovskii, On some Banach lattices IV, Sibirsk. Mat. Z. 14 (1973) 140–155. (Russian); English translation in; Sib. Math. J. 14 (1973) 97–108.
[19] G.Ya. Lozanovskii, Transformations of ideal Banach spaces by means of concave functions, Qualitative and approximate methods for the investigation of operator equations, Yaroslavl' (1978) 122–147. (Russian).
[20] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. 24 (1964) 113–190.
[21] V.A. Shestakov, Transformations of Banach ideal spaces and interpolation of linear operators, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (11–12) (1981) 569–577. 1982. (Russian).
[22] A. Kamińska, Some remarks on Orlicz–Lorentz spaces, Math. Nachr. 147 (1990) 29–38.
[23] D. Cruz-Uribe, SFO, M. Krbec, Localization and extrapolation in Orlicz–Lorentz spaces, in: M. Cwikel, A. Kufner, L. Persson, G. Sparr (Eds.), Function Spaces, Interpolation Theory and Related Topics (Lund 2000), DeGruyter, Berlin, 2002, pp. 273–284.
[24] S.F. Lukomskii, Convergence of Fourier series in Lorentz spaces, East J. on Approx. 9 (2) (2003) 229–238.
[25] M. Mastyło, Lattice structures on some Banach spaces, Proc. Am. Math. Soc. (2011) doi:10.1090/ S0002-9939-2011-11151-1. Article electronically published on 16 August 2011.
[26] A. Aliprantis, O. Burkinshaw, Positive Operators, in: Pure and Applied Mathematics Series, vol. 119, Academic Press, New York–London, 1985.