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RADEMACHER FUNCTIONS IN SYMMETRIC SPACES

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To Asya—for her patience and understanding

CONTENTS

Preface . . . 725
Chapter 0. Definitions, Notation, and Preliminaries . . . 727
1. Banach Ideal Spaces . . . 727
2. Symmetric Spaces . . . 728
3. Definitions and Results from Probability Theory . . . 731
4. Basic Sequences and Lacunary Systems . . . 732
5. Interpolation of Operators and Real Method Spaces . . . 735
Chapter 1. Elementary Properties of Rademacher Functions . . . 737
Chapter 2. Rademacher Sums in L_p-Spaces and Their Distribution . . . 751
Chapter 3. Rademacher System in Symmetric Spaces "Far" from L_∞ . . . 760
Chapter 4. Rademacher Systems in Symmetric Spaces "Close" to L_∞ . . . 769
Chapter 5. Rademacher Functions and Cones of Step Functions . . . 787
Chapter 6. Rademacher Chaos in Symmetric Spaces . . . 798
Chapter 7. Subspace Rad X . . . 819
Chapter 8. Systems Equivalent in Distribution to the Rademacher System . . . 827
Chapter 9. Selection of Lacunary Subsystems . . . 840
Chapter 10. Rademacher Multiplicator Spaces . . . 860
References . . . 881

PREFACE

The Rademacher system or the Bernoulli sequence of independent, identically and symmetrically distributed random variables taking values ± 1 is a classical object of orthogonal series theory and probability theory. It has many applications in other fields as well, first of all, in the geometric theory of Banach spaces, theory of operators, harmonic analysis, and mathematical statistics. The topic of this monograph is more specific: the study of the Rademacher system in symmetric (rearrangement invariant) functional spaces. The first work devoted to this problem (apart from the classical results for L_p-spaces such as the Khintchine inequality) is an article by Rodin and Semenov that was published in 1975. A number of principal problems related to the behavior of Rademacher systems in symmetric spaces was posed and the path for further investigations was indicated. We recall that the initial works on general symmetric spaces (first of all, works of G. Lorentz and Semenov) appeared in 50s-60s of the last century. Then [92, 96, 97] summarized a certain stage of the investigations.

Investigations of the Rademacher systems in symmetric spaces received renewed impetus in 90th after the works by S. Montgomery-Smith and P. Hitchenko, where they refined the Khintchine inequality. The main role was played by the Peetre $\mathcal{K}$-functional. It is one of the main tools in the modern theory of interpolation of operators. Later, in the author's work, it was shown that this theory (the

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real $\mathcal{K}$-method) is a natural and effective tool to describe subspaces generated by the Rademacher system in symmetric spaces and to compare it with general systems of measurable functions (random variables). Perhaps, the main focus of this book is the study of the relation between properties of the Rademacher system and interpolational constructions.

It is necessary to say a few words to support the choice of symmetric spaces as the most natural "field" for the Rademacher system, and to explain that an interesting and quite substantial theory arises in this case. The point is that the Rademacher functions generate a symmetric space of sequences in every symmetric space. As a rule, this sequence has a quite simple description. This description is frequently not unique.

To read the book, one should be familiar with the symmetric space theory and the theory of interpolation of operators. For the convenience of readers, we provide information concerning these theories and other fields of functional analysis and theory of functions in Chap. 0. Then, in Chap. 1, we give classical facts concerning the Rademacher system; they are mainly related to the convergence of series and estimations of L_p-norms for polynomials. Here, the central point is the Khintchine inequality and its variants. In the same chapter we give necessary properties of the Rademacher system. They are used in general symmetric spaces. In Chap. 2, newer results are given. They improve the Khintchine inequality that has not been considered in monographs so far. Here an intensive application of methods of interpolation starts. The main part of Chaps. 3 and 4 is a study of the subspace of a symmetric space generated by the Rademacher system. It turns out that the properties of this subspace is characterized by the degree of "closeness" of the symmetric space to the space L_∞. Chapter 5 shows one more possibility to describe this subspace as a cone of step functions. In Chap. 6, the Rademacher chaos is considered; necessary and sufficient conditions of its unconditionality in a symmetric space are found. In Chap. 7, we consider the subspace of a symmetric space on a square generated by the Rademacher system with vector coefficients and study the properties of that subspace. The next two chapters are less related to the theory of symmetric spaces. But the method of interpolation still plays an important role: Comparison problems for arbitrary systems of measurable functions (random variables) and the Rademacher system are studied. Lacunary subsystem separations are studied as well. These subsystems are equivalent in distribution to the "model" system. Chapter 10 is devoted to the study of the Rademacher multiplicator space. First of all, we are interested when this space is symmetric.

Each section has its own numeration. We use the sign $:=$ to show that the equality is a definition. We denote by $\operatorname{card} E$ the number of elements of a finite set E, and by $[x]$ the integer part of a real number x.

Vector-valued Rademacher series and applications of the Rademacher functions to the geometric theory of Banach spaces are studied. Problems related to both directions are developed intensively and can be a good reason to write a separate book.

Beginning from Chap. 1, each chapter ends with comments and references related to its content. The main role of such comments is to show close research directions. It is clear that not all noteworthy works are mentioned here. We tried to mention only key works, which compensates for this incompleteness.

The material of the initial three sections is the basis of a special course offered at Samara State University in the spring and autumn of 2004. Also, results of the book were reported at many different seminars, schools, and conferences. Results from Chaps. 4 and 10 were partially reported at the conference "Differential equations and their applications" (Samara, 1996), All-Russian Winter Mathematical School (Voronezh, 1997 and 2001), the function theory seminar at Samara State University (Samara, 2002 and 2007), the seminar at Lulea University of Technology (Sweden, 2002), and seminars at Universities of Madrid and Seville (Spain, 2002 and 2003). The author's report relied on facts from Chap. 6 delivered at the Steklov Mathematical Institute seminar under the direction of Corresponding Member of the RAS B. S. Kashin (Moscow, 1997) and at the MSU seminar under the direction of Corresponding Members of the RAS P. L. Ulyanov and B. S. Kashin (Moscow, 1998), at the Conference "Algebra and analysis" (Kazan, 1997), at the international conferences on mathematical analysis

(St. Petersburg, 1998) and "Harmonic Analysis and Approximation" (Armenia, 1998), and at the all-Russian Winter Mathematical School (Voronezh, 2007). Finally, the results of Chap. 8 were reported at the Steklov Mathematical Institute seminar under the direction of Corresponding Member of the RAS B. S. Kashin (Moscow, 1999), at the MSU seminar under the direction of Corresponding Members of the RAS P. L. Ulyanov and B. S. Kashin (Moscow, 1999), at the all-Russian school-colloquium on stochastic methods (Samara, 1999), and at the international conferences on mathematical analysis (St. Petersburg, 1999), "Function Theory, its Applications and Closely Related Questions" (Kazan, 1999), and "Functional Spaces" (Wroclaw, 2001).

I am grateful to my advisor E. M. Semenov and to my co-author and friend F. A. Sukochev for reading the book and making some important critical observations. I would especially like to express my gratitude to my disciple C. V. Lykov who, in fact, was the scientific editor. Due to his selfless labor, we managed to reduce the number of mistakes and misprints.

Chapter 0

DEFINITIONS, NOTATION, AND PRELIMINARIES

1. Banach Ideal Spaces

Let (T, Σ, μ) be a σ-finite measure space μ set on the σ-algebra Σ of subsets of a set T. The set of a.e. finite real-valued functions (equivalence classes) on T with the natural algebraic operation and convergence topology with respect to the measure μ on sets of finite measure is a linear topological space. Denote it by $S(T, \Sigma, \mu)$. As usual, functions coinciding a.e. are identified. The notation $x \le y$ $(x, y \in S(T, \Sigma, \mu))$ means that $x(t) \le y(t)$ for almost all $t \in T$.

Further, we mostly consider the following cases of space $S(T, \Sigma, \mu)$:

(a) $S(I)$, where $I = [0, 1]$ with the Lebesgue measure;
(b) $S(I \times I)$ with the Lebesgue measure;
(c) $S(\mathbb{Z})$, $\mu(\{k\}) = 1$ $(k \in \mathbb{Z})$;
(d) $S(\mathbb{N})$, $\mu(\{k\}) = 1$ $(k \in \mathbb{N})$;
(e) $S(0, \infty)$ with the Lebesgue measure.

In the sequel, we denote by $m(B)$ the Lebesgue measure of the set $B \subset \mathbb{R}^n$ and by $\chi_B(t)$ the characteristic function of the set $B \subset T$, i.e.,

$$\chi_B(t) = \begin{cases} 1, & t \in B \\ 0, & t \notin B. \end{cases}$$

A linear subspace $E \subset S(T, \Sigma, \mu)$ is called *ideal* if for all $y \in E$ the set $\{x \in S(T, \Sigma, \mu) : |x| \le |y|\}$ is a subset of E. Any ideal space E with a *monotone* norm (i.e., $\|x\| \le \|y\|$ for any $x, y \in E$ such that $|x| \le |y|$) is called a *normalized ideal* space. Moreover, if such a space is complete, then it is called a *Banach ideal* space (*Banach lattice*).

Any Banach ideal space E over a measure space (T, Σ, μ) is linearly and continuously embedded into a separable linear topological space $S(T, \Sigma, \mu)$ (see [84, Theorem 4.3.1]). In particular, this means that the convergence $\|x_n - x\|_E \to 0$ $(x_n, x \in E)$ implies the convergence $x_n \to x$ with respect to the measure μ on sets of finite measure from T.

If E is a Banach ideal space, then an *associate* (or *dual*) space E' consists of all $y \in S(T, \Sigma, \mu)$ such that

$$\|y\|_{E'} \;=\; \sup\left\{ \int_T x(t)y(t)\, d\mu : \;\; \|x\|_E \le 1 \right\} \;<\; \infty$$

(it is clear that it is necessary to substitute integrals for sums in the case of sequence spaces). The space E' is isometrically embedded into the adjoint space E^*. Any Banach ideal space E is continuously embedded into its second associate. Denote it by E''. A norm in a Banach ideal space is called *order semi-continuous* if from $x_n \in E$ $(n = 1, 2, \dots)$, $x \in E$, and $x_n \to x$ a.e. on T, it follows that $\|x\|_E \le \liminf_{n\to\infty} \|x_n\|_E$. A norm in E is order semi-continuous if and only if E is embedded into E'' isometrically (see [84, Theorem 6.1.6]), i.e.,

$$\|x\|_E = \sup\left\{ \int_T x(t)y(t)\,d\mu : \ \|y\|_{E'} \le 1 \right\}.$$

A Banach ideal space E is called *maximal* (has the *Fatou property*) if from $x_n \in E$ $(n = 1, 2, \dots)$, $\sup_{n=1,2,\dots} \|x_n\|_E < \infty$, $x \in S(T, \Sigma, \mu)$ and $x_n \to x$ a.e. on T, it follows that $x \in E$ and

$$\|x\|_E \le \liminf_{n\to\infty} \|x_n\|_E.$$

A space E is maximal if and only if its natural embedding to E'' is a surjective isometry (otherwise $E = E''$) (see [84, Theorem 6.1.7]). We say that an element $x \in E$ has an *absolutely continuous* norm in the Banach ideal space E if for a monotonely decreasing sequence of measurable sets $B_n \subset T$ $(n = 1, 2, \dots)$ such that $\bigcap_{n=1}^{\infty} B_n = \varnothing$, the following condition holds: $\|x \cdot \chi_{B_n}\|_E \to 0$. If E is a Banach ideal space over one of the above spaces (a)–(e), then it is separable if and only if any $x \in E$ has an absolutely continuous norm. Moreover, each of these conditions is equivalent to the (isometric) coincidence of the dual space E' with the adjoint spaced E^*. A detailed description of the Banach ideal spaces theory is provided in [39, 84].

2. Symmetric Spaces

Symmetric spaces represent an important case of Banach ideal spaces. These spaces could be considered as a natural generalization of p-summable functions. The theory of symmetric (commutatively invariant) spaces is the subject of monographs [39, 92, 97]; we recall here some necessary definitions and results. These results are mostly related to spaces of functions set on $[0, 1]$.

For a function $x = x(t) \in S[0, 1]$, $x \ge 0$, introduce the distribution function $n_x(\tau)$:

$$n_x(\tau) = m\{t \in [0, 1] : \ x(t) > \tau\} \ \ (\tau > 0)$$

It is nonnegative, decreasing, and right continuous. The nonnegative functions $x \in S[0, 1]$ and $y \in S[0, 1]$ are called equimeasurable if $n_x(\tau) = n_y(\tau)$ for all $\tau > 0$. In particular, any function $x \in S[0, 1]$, $x \ge 0$, is equimeasurable with its decreasing, left continuous rearrangement

$$x^*(t) = \inf\{\tau \ge 0 \ : \ n_x(\tau) < t\} \ \ (0 < t \le 1).$$

Denote by $x^*(t)$ a rearrangement of the function $|x(t)|$ for an arbitrary function $x(t) \in S[0, 1]$.

Definition 0.1. A Banach ideal space $X \subset S[0, 1]$ is called *symmetric* if from the fact that $x \in X$, $y \in S[0, 1]$, and functions $|x|$ and $|y|$ are equimeasurable, it follows that $y \in X$ and $\|y\|_X = \|x\|_X$.

Examples of symmetric spaces. First, they are L_p $(1 \le p \le \infty)$ spaces:

$$\|x\|_p = \begin{cases} \left(\int\limits_0^1 |x(t)|^p\,dt \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}\limits_{t\in[0,1]} |x(t)|, & p = \infty \end{cases}$$

and (their generalization) the family of spaces $L_{p,q}$ $(1 < p < \infty,\ 1 \le q \le \infty)$:

$$\|x\|_{p,q} = \begin{cases} \left[\dfrac{q}{p} \displaystyle\int_0^1 \left(t^{1/p} x^*(t)\right)^q \frac{dt}{t} \right]^{1/q}, & 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{t \in [0,1]} \left(t^{1/p} x^*(t)\right), & q = \infty. \end{cases}$$

These spaces are important in the theory of interpolation of operators. Though functional $\|x\|_{p,q}$ is non-subadditive, it is equivalent to the norm $\|x\|'_{p,q} = \|x^{**}\|_{p,q}$, where

$$x^{**}(t) = \frac{1}{t} \int_0^t x^*(s)\, ds.$$

One can easily check [92, Lemma 2.6.5] that $L_{p,q_1} \subset L_{p,q_2}$ $(1 \le q_1 \le q_2 \le \infty)$ and $L_{p,p} = L_p$.

Another natural generalization of L_p-spaces are Orlicz spaces (see [90, 102]). If $M(u)$ is an increasing convex function on $[0, \infty)$, $M(0) = 0$, then the *Orlicz space* L_M consist of all $x \in S[0,1]$ such that there exists $\lambda > 0$ such that

$$\int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt \le 1.$$

In the sequel, in this space we use the *Luxemburg norm* $\|x\|_{L_M} = \inf \lambda$, where the infimum is taken over all λ such that the latter inequality holds. The space $(L_M)'$ associated to L_M coincides with the Orlicz space L_{M^*} constructed on the function $M^*(u)$. This function is complementary (in the Jung sense) to $M(u)$, i.e.,

$$M^*(u) = \sup_{v \ge 0} (uv - M(v))$$

(see [90, Sec. 14]).

In the sequel, one functional class plays a prominent role. Recall that a nonnegative function $\varphi(t)$ defined in $t \ge 0$ is called *quasiconcave* if $\varphi(0) = 0$, $\varphi(t)$ increases and $\varphi(t)/t$ decreases. In particular, any function nonnegative concave on semi-axis $(0, \infty)$ redefined by zero at the origin is quasiconcave (see [92, Sec. 2.1, p. 67]). On the other hand, any function $\varphi(t)$ quasiconcave on the set $[0, \infty)$ is equivalent to the least concave majorant $\bar{\varphi}(t)$ (see [92, Theorem 2.1.1]):

$$\frac{1}{2}\bar{\varphi}(t) \le \varphi(t) \le \bar{\varphi}(t) \ \ (t > 0).$$

If φ is a quasiconcave function and $p \in [1, \infty)$, then the *Lorentz space* $\Lambda_p(\varphi)$ consists of all $x \in S[0,1]$ such that

$$\|x\|_{\varphi,p} = \left(\int_0^1 (x^*(t))^p \, d\bar{\varphi}(t) \right)^{1/p} < \infty,$$

where $\bar{\varphi}$ is the least concave majorant φ. In particular, $\Lambda(\varphi) := \Lambda_1(\varphi)$. The *Marcinkiewicz space* $M(\varphi)$ is the set of all $x \in S[0,1]$ such that

$$\|x\|_{M(\varphi)} = \sup_{0 < t \le 1} \frac{1}{\varphi(t)} \int_0^t x^*(s)\, ds < \infty.$$

The Lorentz space and Marcinkiewicz space are associates to each other, i.e., $(\Lambda(\varphi))' = M(\varphi)$ and $(M(\varphi))' = \Lambda(\varphi)$ (see [92, Theorems 2.5.2 and 2.5.4]).

If X and Y are Banach spaces, then the notation $X \subset Y$ means that X is linearly and continuously embedded into Y, i.e., from $x \in X$, it follows that $x \in Y$, the structure X as a subspace of Y coincides with the linear subspace of Y, and there exists a constant $C > 0$ such that $\|x\|_Y \le C\|x\|_X$ for all $x \in X$. One can easily check (see [92, Theorem 2.4.1]) that among all symmetric spaces on $[0,1]$ with respect to the given order, there is the greatest and the least symmetric space. More exactly, for an arbitrary symmetric space X, we have

$$L_\infty \subset X \subset L_1.$$

Denote by X° the *separable part* of symmetric space X, i.e., the closure of L_∞ in X. Note that X° is a symmetric space, which is separable only if $X \neq L_\infty$ [92, Lemma 2.4.4]. In particular, the space $M(\varphi)^\circ$ can be characterized as the set of functions $x(t)$ such that

$$\lim_{t\to 0+} \frac{1}{\varphi(t)} \int\limits_0^t x^*(s)\,ds = 0.$$

A *fundamental function* $\varphi_X(s) := \|\chi_{(0,s)}\|_X$ $(0 \le s \le 1)$ is an important characteristic of the symmetric space. One can easily check that

$$\varphi_{L_{p,q}}(s) = s^{1/p}\,(1 \le q \le \infty),\ \varphi_{\Lambda(\varphi)}(s) = \bar{\varphi}(s) \ \text{ and }\ \varphi_{M(\varphi)}(s) = \tilde{\varphi}(s), \ \ \text{where}\ \ \tilde{\varphi}(s) = s/\varphi(s).$$

Moreover,

$$\varphi_{L_M}(s) = \frac{1}{M^{-1}(1/s)} \qquad (M^{-1} \text{ is the function inverse to } M).$$

For any symmetric space, φ_X is a quasiconcave function on $[0,1]$ (see [92, Theorem 2.4.7]) and the Lorentz space and Marcinkiewicz space are extreme in a class of symmetric spaces with the same fundamental function (see [92, Theorems 2.5.5 and 2.5.7]). If φ is a fundamental function of symmetric space X, then

$$\Lambda(\varphi) \subset X \subset M(\tilde{\varphi}).$$

The *dilation function* $\mathcal{M}_f(t)$ of a function f set on $(0,l)$, where $l = 1$ or $l = \infty$, is defined as follows:

$$\mathcal{M}_f(t) = \sup\left\{\frac{f(st)}{f(s)} : s \in (0, l\min(1,1/t))\right\}, \quad t > 0.$$

Since $\mathcal{M}_f$ is semimultiplicative (i.e., $\mathcal{M}_f(t_1t_2) \le \mathcal{M}_f(t_1)\cdot\mathcal{M}_f(t_2)$ for arbitrary $t_1 > 0$ and $t_2 > 0$), it follows that there exist numbers

$$\gamma_f = \lim_{t\to 0+} \frac{\ln \mathcal{M}_f(t)}{\ln t} = \sup_{0<t<1} \frac{\ln \mathcal{M}_f(t)}{\ln t}$$

and

$$\delta_f = \lim_{t\to +\infty} \frac{\ln \mathcal{M}_f(t)}{\ln t} = \inf_{1<t<\infty} \frac{\ln \mathcal{M}_f(t)}{\ln t}$$

called the *lower and upper dilation indices* of the function f [92, Sec 2.1, p. 75-76] respectively. If φ is a quasiconcave function, then $0 \le \gamma_\varphi \le \delta_\varphi \le 1$.

A *dilation operator*

$$\sigma_\tau x(t) := x(t/\tau)\cdot\chi_{[0,\min(1,\tau)]}(t) \ \ (\tau > 0)$$

is continuous in each symmetric space X and $\|\sigma_\tau\|_{X\to X} \le \max(1,\tau)$ (see [92, Theorem 2.4.5]). With its help, the *lower and upper Boyd indices*

$$\alpha_X = \lim_{\tau\to 0+} \frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln \tau} = \sup_{0<\tau<1} \frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln \tau}$$

and

$$\beta_X = \lim_{\tau\to +\infty} \frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln \tau} = \inf_{\tau>1} \frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln \tau}$$

of a space X are defined. For any symmetric space X, we have

$$0 \le \alpha_X \le \gamma_{\varphi_X} \le \delta_{\varphi_X} \le \beta_X \le 1,$$

where γ_{φ_X} and δ_{φ_X} are the lower and upper dilation indices of the fundamental function (see [92, Sec. 2.1, p. 138]). Also, one can check that

$$\alpha_{L_{p,q}} = \beta_{L_{p,q}} = 1/p,\ \alpha_{\Lambda(\varphi)} = \gamma_\varphi\,,\ \beta_{\Lambda(\varphi)} = \delta_\varphi,\ \alpha_{M(\varphi)} = \gamma_{\tilde{\varphi}}\,,\ \beta_{M(\varphi)} = \delta_{\tilde{\varphi}}.$$

Further, we need the concept of symmetric space of random variables defined on the probability space $(\Omega, \Sigma, \mathbb{P})$. More exactly, if X is a symmetric space on $[0,1]$, then the respective space $X(\Omega) := X(\Omega, \Sigma, \mathbb{P})$ consists of all random variables $f : \Omega \to \mathbb{R}$ such that $f^* \in X$. Here $f^*(t)$ is a decreasing left continuous transposition of the random variable f, i.e.,

$$f^*(t) = \inf\{\tau \geq 0\ :\ \mathbb{P}\{\omega \in \Omega :\ |f(\omega)| > \tau\} < t\}\ \ (0 < t \leq 1).$$

Moreover, $\|f\|_{X(\Omega)} := \|f^*\|_X$. This definition allows us to transfer all the remaining concepts of symmetric spaces theory to the case of $X(\Omega)$.

In particular, denote the norm $\|f\|_{L_p(\Omega)}$ $(1 \leq p \leq \infty)$ by $\|f\|_p$. The case where Ω is the square $I \times I$ with the Lebesgue measure is considered in more details.

Symmetric spaces of sequences are considered less. They can be defined as follows. A Banach ideal space of sequences E is called symmetric if from the fact that $a = (a_k)_{k=1}^\infty \in E$ and $\pi : \mathbb{N} \to \mathbb{N}$ is a random permutation, it follows that the "transposed" sequence $a_\pi = (a_{\pi(k)})_{k=1}^\infty$ belongs to E and $\|a_\pi\|_E = \|a\|_E$. Apart from spaces ℓ_p $(1 \leq p \leq \infty)$,

$$\|a\|_p = \begin{cases} \left(\sum\limits_{k=1}^{\infty} |a_k|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \sup\limits_{k=1,2,\ldots} |a_k|, & p = \infty, \end{cases}$$

Marcinkiewicz spaces $m(\varphi)$, where φ is a quasiconcave function on $[0, \infty)$, are examples of symmetric spaces, $m(\varphi)$, and they consist of all bounded sequences of real numbers $a = (a_k)_{k=1}^\infty$ such that

$$\|a\|_{m(\varphi)} = \sup_{n=1,2,\ldots} \frac{1}{\varphi(n)} \sum\nolimits_{k=1}^{n} a_k^* < \infty,$$

where $(a_k^*)_{k=1}^\infty$ is a decreasing permutation of the sequence $(|a_k|)_{k=1}^\infty$.

Nonnegative functions $f(t)$ and $g(t)$ are called equivalent on the set T (we write: $f \asymp g$ $(t \in T)$) if there exists a constant $C > 0$ such that the following condition holds fof all $t \in T$:

$$C^{-1} f(t) \leq g(t) \leq C\, f(t).$$

Norm (quasinorm) equivalence is defined in the same way.

3. Definitions and Results from Probability Theory

Definition 0.2. A set of random variables $\{f_k\}_{k=1}^n$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ is called *independent* if for any intervals I_k on $\mathbb{R}$, the following relation holds:

$$\mathbb{P}\{\omega \in \Omega :\ f_k(\omega) \in I_k,\ k = 1, 2, \ldots, n\} = \prod_{k=1}^{n} \mathbb{P}\{\omega \in \Omega :\ f_k(\omega) \in I_k\}.$$

A sequence of random variables $\{f_k\}_{k=1}^\infty$ is called *independent* if for every $n \in \mathbb{N}$, the set $\{f_k\}_{k=1}^n$ is independent.

In the sequel, the following standard version of the central limit theorem is used (see [46, Theorem 7.2]). Let $\{\xi_k\}_{k=1}^\infty$ be the sequence of independent identically distributed variables defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ with expectation a and dispersion σ^2, $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$, and

$$U_n = \frac{S_n - an}{\sigma\sqrt{n}}.$$

Then

$$\lim_{n\to\infty} \mathbb{P}\{\omega \in \Omega :\, U_n(\omega) < x\} = \Phi(x)$$

for any $x \in \mathbb{R}$, where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-t^2/2}\, dt$.

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, and f be an integrable random variable set on it. Then if $\Re$ is a σ-subalgebra of the σ-algebra Σ, then there exists a unique (up to zero measure) $\Re$-measurable integrable random variable $\mathbb{E}[f|\,\Re]$ such that

$$\int\limits_B f(\omega)\, d\mathbb{P}(\omega) = \int\limits_B \mathbb{E}[f|\,\Re](\omega)\, d\mathbb{P}(\omega)$$

for all $B \in \Sigma$ (see [65, Theorem 1.1]). A variable $\mathbb{E}[f|\,\Re]$ is called the *conditional expectation* of f with respect to the σ-algebra $\Re$. If $\Re$ is a σ-subalgebra generated by a random variables $g_1, g_2, \dots, g_n$, then the conditional expectation with respect to it is denote by $\mathbb{E}[f|\, g_1, \dots, g_n]$. Further, we will use the following notation: $\mathbb{E}[f] = \int\limits_\Omega f(\omega)\, d\mathbb{P}(\omega)$.

Finally, we recall the definition of a martingale with respect to an increasing sequence of σ-algebras.

Definition 0.3. Let $\{\Re_i\}_{i\in I}$ be a finite or infinite sequence of σ-subalgebras of a σ-algebra Σ such that $\Re_i \subset \Re_j$ if $i, j \in I$ and $i \le j$. A sequence of random variables $\{\xi_i\}_{i\in I}$ is called a *martingale* with respect to $\{\Re_i\}_{i\in I}$ if

(i) for every $i \in I$, ξ_i is $\Re_i$-measurable and integrable on Ω;
(ii) if $i, j \in I$ and $i \le j$, then $\mathbb{E}[\xi_j|\,\Re_i] = \xi_i$.

For detailed descriptions, see, e.g., in [46, 65].

4. Basic Sequences and Lacunary Systems

Let X be a Banach space over a field of real numbers.

Definition 0.4. A sequence $\{x_n\}_{n=1}^\infty \subset X$ is called *basic* if it is a basis in a closed linear span $[x_n]_X$, i.e., any $x \in [x_n]_X$ is uniquely defined as

$$x = \sum_{i=1}^\infty a_i x_i, \quad \text{where} \quad a_i \in \mathbb{R}.$$

It is well known (see, e.g., [96, Proposition 1.a.3]) that the sequence $\{x_n\}_{n=1}^\infty \subset X$ is basic if and only if the following conditions hold:

(1) $x_n \neq 0$ for all $n \in \mathbb{N}$;
(2) The family of projectors P_m $(m = 1, 2, \dots)$ such that $P_m(x_i) = x_i$ $(i \le m)$ and $P_m(x_i) = 0$ $(i > m)$ is uniformly bounded on $[x_n]_X$.

The major sense of the second condition is as follows: there exists a constant $K > 0$ such that for any natural $m < n$ and $a_i \in \mathbb{R}$, we have

$$\Big\|\sum_{i=1}^m a_i x_i\Big\| \le K \Big\|\sum_{i=1}^n a_i x_i\Big\|.$$

Definition 0.5. A basic sequence $\{x_n\}_{n=1}^\infty$ of a Banach space is called *unconditional* if for any permutation π of the positive integers, the sequence $\{x_{\pi(n)}\}_{n=1}^\infty$ is basic in X as well

One finds easily (see also [86, theorem 1.1.1] or [96, Proposition 1.c.1]) that a basic sequence $\{x_n\}_{n=1}^\infty$ is unconditional if and only if from the convergence of a series $\sum\limits_{i=1}^\infty a_i x_i$ $(a_i \in \mathbb{R})$ follows the convergence of the series $\sum\limits_{i=1}^{\infty} \theta_i a_i x_i$, where $\theta = \{\theta_i\}_{i=1}^{\infty}$ is an arbitrary sequence of signs, i.e., $\theta_i = \pm 1$. According to the closed graph theorem and the uniform boundedness principle of the Banach–Steinhaus uniform boundedness principle, the unconditionality of $\{x_n\}_{n=1}^{\infty}$ is equivalent to the uniform boundedness of the family of sign–ordering operators

$$M_\theta \left(\sum_{i=1}^{\infty} a_i x_i \right) = \sum_{i=1}^{\infty} \theta_i a_i x_i \quad (\theta_i = \pm 1)$$

defined on $[x_n]_X$. In other words, $\{x_n\}_{n=1}^{\infty}$ is unconditional if and only if there exists a constant K_0 such that for arbitrary $n \in \mathbb{N}$, signs θ_i, and $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots)$, the following inequality holds:

$$\Big\| \sum_{i=1}^{n} \theta_i a_i x_i \Big\| \le K_0 \Big\| \sum_{i=1}^{n} a_i x_i \Big\|.$$

The concept of a system of stochastic unconditional convergence is a generalization of the latter concept. Before we give an equivalent definition based on the using of the Rademacher system, we recall that the system (x_n, x_n^*), $x_n \in X$, $x_n^* \in X^*$ $(n = 1, 2, \ldots)$ is called *biorthogonal* if $x_m^*(x_n) = 0$ for $m \ne n$.

Definition 0.6. Let X be a Banach Space and X^* be its dual space. A biorthogonal system (x_n, x_n^*), $x_n \in X$, $x_n^* \in X^*$ $(n = 1, 2, \ldots)$, is called a *system of random unconditional convergence* (*RUC-System*) if there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $x \in [x_n]_X$, the following inequality holds:

$$\int_0^1 \Big\| \sum_{i=1}^{n} r_i(s) x_i^*(x) x_i \Big\|_X \, ds \le C \|x\|_X.$$

In particular, if $\{x_n\}_{n=1}^{\infty}$ is an orthogonal system of bounded functions on $[0, 1]$, then (x_n, x_n) is biorthogonal in any symmetric space. In this case, we say that RUC-system is the sequence $\{x_n\}$.

Definition 0.7. A basic sequence $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is called *symmetric* if it is equivalent to any sequence in the form $\{x_{\pi(n)}\}_{n=1}^{\infty}$ in X, where π is a permutation of the positive integers, i.e., there exists $C > 0$ such that for arbitrary real a_n $(n = 1, 2, \ldots)$, the following inequalities hold:

$$C^{-1} \Big\| \sum_{n=1}^{\infty} a_n x_n \Big\|_X \le \Big\| \sum_{n=1}^{\infty} a_n x_{\pi(n)} \Big\|_X \le C \Big\| \sum_{n=1}^{\infty} a_n x_n \Big\|_X.$$

It follows from the definitions that any symmetric sequence is unconditional as well. More detailed information about basis, basic sequences, and their properties is provided in [96, Chaps. 1, 3] and [86, Chap. 1].

We are interested in so-called lacunary systems of functions (or random variables). The term "lacunarity" means that their behavior in various spaces is in a way similar to the behavior of a system of independent functions. One can say that they are "rarefied" sequences, which are not complete in the space where they are considered. We recall that a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of a Banach space X is *complete* if its closed span $[x_n]_X$ coincides with the whole X. The main object of the investigation of this book (the sequence of Rademacher functions $r_n(t) = \operatorname{sign} \sin 2^n \pi t$ $(t \in [0, 1])$, $n = 1, 2, \ldots,$) is a brilliant representative of lacunary systems. As we will see in the next section, this system is not only orthogonal but strongly multiplicative in the sense of the following definition.

Definition 0.8. A finite or countable system of random variables $\{f_n\}$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ is called *multiplicative* if for arbitrary pairwise different $n_1, n_2, \ldots, n_k$ $(k \in \mathbb{N})$, the following relation holds:

$$\mathbb{E}(f_{n_1} \cdot f_{n_2} \ldots f_{n_k}) = 0.$$

If

$$\mathbb{E}(f_{n_1} \cdot f_{n_2} \dots f_{n_k} \cdot f_n^2) = 0$$

for any pairwise different $n_1, n_2, \dots, n_k$ and $n \neq n_s$ $(s = 1, 2, \dots, k)$, then the system $\{f_n\}$ is called *strongly multiplicative*.

Very important examples of such systems are provided below.

Example 0.1 (Lacunary trigonometric system)**.** Let $f_n(x) = \sin(2\pi k_n x)$, where $(k_n)_{n=1}^\infty$ is an increasing sequence of integers and $x \in [0, 1]$. One can easily see that this system is multiplicative if $k_{n+1}/k_n \geq 2$ and strongly multiplicative if $k_{n+1}/k_n \geq 3$. Note that the difference between those conditions is important because, e.g., the system $\{\sin(2^n \pi x)\}_{n=1}^\infty$ is multiplicative but not strongly multiplicative. Indeed, decreasing the degree, we obtain

$$\int_0^1 \sin^2(2\pi x)\sin(4\pi x)\sin(8\pi x)\,dx = \frac{1}{2}\Bigg(\int_0^1 \sin(4\pi x)\sin(8\pi x)\,dx - \int_0^1 \cos(4\pi x)\sin(4\pi x)\sin(8\pi x)\,dx\Bigg) = -\frac{1}{4}\int_0^1 \sin^2(8\pi x)\,dx < 0.$$

Example 0.2 (Independent random variables with mean zero variables)**.** A finite or countable sequence $\{f_n\}$ of independent random variables such that $f_n \in L_2 = L_2(\Omega, \Sigma, \mathbb{P})$ and $\mathbb{E}f_n = 0$ $(n = 1, 2, \dots)$ forms a strongly multiplicative system. It is a consequence of the relation

$$\mathbb{E}(g_1 \cdot g_2 \cdot \dots \cdot g_k) = \mathbb{E}g_1 \cdot \mathbb{E}g_2 \dots \mathbb{E}g_k,$$

which holds for any family of independent functions $g_1, g_2, \dots, g_k$ (see [86, Theorem 2.2.4]).

Example 0.3 (Martingale differences)**.** Let $\{\xi_i\}$ be a finite or countable martingale with respect to an increasing sequence of σ-algebras $\{\mathfrak{F}_i\}$. Then the corresponding system of martingale differences $f_i := \xi_i - \xi_{i-1}$ $(i = 1, 2, \dots)$, where $\xi_0 := \mathbb{E}\xi_1$, is multiplicative. Indeed, if $i_1 < i_2 < \dots < i_k$, then, due to conditional expectation properties (see [65, Lemma 1.13]), we have

$$\int_\Omega f_{i_1} \dots f_{i_k}\,d\mathbb{P} = \int_\Omega \mathbb{E}\big(f_{i_1} \cdot \ldots \cdot f_{i_k} | \mathfrak{F}_{i_k - 1}\big)\,d\mathbb{P} = \int_\Omega f_{i_1} \cdot \ldots \cdot f_{i_{k-1}} \cdot \mathbb{E}\big(f_{i_k} | \mathfrak{F}_{i_k - 1}\big)\,d\mathbb{P} = 0$$

because $\mathbb{E}\big(f_{i_k} | \mathfrak{F}_{i_k - 1}\big) = \mathbb{E}\big(\xi_{i_k} | \mathfrak{F}_{i_k - 1}\big) - \xi_{i_k - 1} = 0$ by the definition of martingales.

As was mentioned, the measure of system lacunarity is the presence of its properties close to properties of systems of independent functions. We provide definitions of the most important ones.

Definition 0.9. A sequence of random variables $\{f_n\}_{n=1}^\infty$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ is called a *Sidon system* if

$$C^{-1}\sum_{n=1}^m |a_n| \leq \Big\|\sum_{n=1}^m a_n f_n\Big\|_\infty \leq C\sum_{n=1}^m |a_n|,$$

where the constant $C > 0$ does not depend on $m \in \mathbb{N}$ and $a_n \in \mathbb{R}$ $(n = 1, 2, \dots, m)$.

Definition 0.10. A system $\{f_n\}_{n=1}^\infty$ of functions measurable on $[0, 1]$ is called a *Sidon–Zygmund system* if there exists a set $E \subset [0, 1]$ with the Lebesgue measure $m(E) > 0$ such that for any interval I, $m(I \cap E) > 0$, from the condition $\sum\limits_{n=1}^\infty a_n f_n(x) \in L_\infty(I)$ it follows that $(a_n)_{n=1}^\infty \in \ell_1$. If we assume in this definition that $E = [0, 1]$, then we say that $\{f_n\}_{n=1}^\infty$ is a *Sidon–Zygmund system* in the narrow sense.

It follows from the definitions that these properties are related to the absolute convergence of lacunary series. An explanation for the name is that the initial reason for their introduction was the classical Sidon theorem on the absolute convergence of lacunary (in the Hadamard sense) trigonometric Fourier series of a bounded function (see [130] and comments to Chap. 10).

Another lacunary property is related to the integrability of sums of series; its discovery is related both to Khintchine inequalities (see Chap. 1) and a similar Zygmund result for lacunary trigonometric series (see [140]).

Definition 0.11. A sequence of random variables $\{f_n\}_{n=1}^{\infty}$, $f_n \in L_p$ $(p > 2)$, is called a *lacunary system of order* p (*S_p-system*) if

$$\Big\|\sum_{n=1}^{m} a_n f_n\Big\|_p \le K_p \Big\|\sum_{n=1}^{m} a_n f_n\Big\|_2,$$

where the constant $K_p > 0$ does not depend on $m \in \mathbb{N}$ and $a_n \in \mathbb{R}$ $(n = 1, 2, \dots, m)$. If $\{f_n\}$ is an S_p-system for any $p > 2$, then it is called an *S_∞-system.*

More detailed information on a lacunary system can be found in [70, 81, 86].

5. Interpolation of Operators and Real Method Spaces

A combination $\vec{X} = (X_0, X_1)$ of two Banach spaces X_0 and X_1 is called a *Banach couple* if they are continuously embedded into the same separable linear topological space. For a Banach couple, we can define the *intersection* $X_0 \cap X_1$ and the *sum* $X_0 + X_1$ as Banach spaces with the norms

$$\|x\|_{X_0 \cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$$

and

$$\|x\|_{X_0 + X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i, i = 0, 1\},$$

respectively. For example, any two Banach ideal spaces E_0 and E_1 defined over the same space with the measure (T, Σ, μ) generate a Banach couple because $E_i \subset S(T, \Sigma, \mu)$ $(i = 0, 1)$.

For any two Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$, denote by $L(\vec{X}, \vec{Y})$ the Banach space of all linear operators $T : X_0 + X_1 \to Y_0 + Y_1$ continuously mapping X_i to Y_i $(i = 0, 1)$ with the norm

$$\|T\|_{L(\vec{X}, \vec{Y})} = \max_{i=0,1} \|T\|_{X_i \to Y_i}.$$

A Banach space X is called *intermediate* for a couple $\vec{X} = (X_0, X_1)$ if $X_0 \cap X_1 \subset X \subset X_0 + X_1$; then the combination (X_0, X_1, X) is called a *triple.* A triple (X_0, X_1, X) is *interpolar* with respect to a triple (Y_0, Y_1, Y) if $T : X \to Y$ for any $T \in L(\vec{X}, \vec{Y})$. Then, due to the closed graph theorem, there exists $C > 0$ such that for any $T \in L(\vec{X}, \vec{Y})$, we have

$$\|T\|_{X \to Y} \le C \|T\|_{L(\vec{X}, \vec{Y})}.$$

If we can assume $C = 1$ in the latter inequality, then the triple (X_0, X_1, X) is called an *exact interpolation* with respect to the triple (Y_0, Y_1, Y). In particular, in the case of $X_0 = Y_0$, $X_1 = Y_1$ and $X = Y$ the space X is called *exactly interpolational* with respect to the couple (X_0, X_1) (or between the spaces X_0 and X_1). Then we write: $X \in \mathcal{I}(X_0, X_1)$.

As an example, we give the classical Riesz–Thorin theorem (see [40, Theorem 1.1.1]). Suppose that (T, μ) and (T', μ') are two spaces with the σ-finite measure, $1 \le p_0, p_1, q_0, q_1 \le \infty$, $p_0 \ne p_1$, and $q_0 \ne q_1$. Then, if all the spaces are considered over a complex field and there exists $\theta \in [0, 1]$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then the triple $(L_{p_0}(T,\mu), L_{p_1}(T,\mu), L_p(T,\mu))$ is an exact interpolation with respect to the triple $(L_{q_0}(T',\mu'),\ L_{q_1}(T',\mu'),\ L_q(T',\mu'))$. If the real number field is studied, then we can guarantee only that the following inequality holds:

$$\|T\|_{L_p(T,\mu)\to L_q(T',\mu')} \le 2\cdot \|T\|_{L(\vec{X},\vec{Y})},$$

here $\vec{X} = (L_{p_0}(T,\mu), L_{p_1}(T,\mu))$, and $\vec{Y} = (L_{q_0}(T',\mu'), L_{q_1}(T',\mu'))$ (see [91, Sec. 1.2, p. 38]).

One of the most important ways to construct interpolation spaces is to use the *Peetre $\mathcal{K}$-functional* defined on the Banach couple (X_0, X_1) as follows:

$$\mathcal{K}(t,x;X_0,X_1) \ = \ \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : \ x = x_0 + x_1, x_i \in X_i\},$$

where $x \in X_0 + X_1$ and $t > 0$. One can easily show that for any Banach couple with fixed $x \in X_0 + X_1$, $\mathcal{K}(t,x;X_0,X_1)$ is a continuous non-negative increasing concave function of the variable $t > 0$ (see [40, Lemma 3.1.1]). On the other hand, for every fixed $t > 0$, it is the norm on the sum $x \in X_0 + X_1$, and all these norms are pairwise equivalent. Besides, if $X_0 \subset X_1$ and $\|x\|_{X_1} \le \|x\|_{X_0}$ $(x \in X_0)$ $(X_1 \subset X_0$ and $\|x\|_{X_0} \le \|x\|_{X_1}$ $(x \in X_1)$ respectively), then $\mathcal{K}(t,x;X_0,X_1) = t\|x\|_{X_1}$ for $0 \le t \le 1$ $(\mathcal{K}(t,x;X_0,X_1) = \|x\|_{X_0}$ as $t \ge 1$ respectively).

Sometimes, the $\mathcal{K}$-functional admits a quite simple equivalent expression. In particular, for an arbitrary space with a σ-finite measure and for any $p \ge 1$, we have

$$\mathcal{K}(t,x;L_p,L_\infty) \asymp \left(\int\limits_0^{t^p} (x^*(s))^p\, ds\right)^{1/p} \quad (t > 0),$$

where the equivalency constants do not depend on $x \in L_p + L_\infty$ and $t > 0$ (for $p = 1$, we can write "=" instead of "$\asymp$"). In further investigations of the Rademacher system in symmetric spaces, a very important role is played by the $\mathcal{K}$-functional corresponding to the Banach couple (ℓ_1, ℓ_2). Denote this functional by $\kappa_a(t)$ $(a = (a_k)_{k=1}^\infty,\ t > 0)$. In Chap. 4, we get equivalent expressions for that, which will be used throughout the whole book.

Then, since F is a Banach ideal space of two-sided real sequences, we denote by $F(u_k)$, $u_k > 0$ $(k = 0, \pm1, \pm2, \ldots)$, a *weighted space* consisting of all sequences $a = (a_k)_{k=-\infty}^\infty$ with the finite norm

$$\|a\|_{F(u_k)} = \|(a_k u_k)\|_F$$

(a functional weighted space is defined in the same way). An arbitrary Banach ideal space of two-sided real sequences F such that $F \supset l_\infty \cap l_\infty(2^{-k})$ is called a *parameter of the real method of interpolation.* If (X_0, X_1) is a Banach couple, then the *space of the real $\mathcal{K}$-method* $(X_0,X_1)_F^{\mathcal{K}}$ consists of all $x \in X_0 + X_1$ such that

$$(\mathcal{K}(2^k,x;X_0,X_1))_k \in F$$

with the norm

$$\|x\| = \|(\mathcal{K}(2^k,x;X_0,X_1))_k\|_F.$$

One can easily check that $(X_0,X_1)_F^{\mathcal{K}}$ is an exact interpolation space between X_0 and X_1 (see [50, Proposition 1.4.2]). In the particular case where $F = \ell_p(2^{-k\theta})$ $(0 < \theta < 1, 1 \le p \le \infty)$, we obtain the classical spaces $(X_0,X_1)_{\theta,p}$ with the norm

$$\|x\|_{(X_0,X_1)_{\theta,p}} = \Big(\sum_{k=-\infty}^{\infty} \big(\mathcal{K}(2^k,x;X_0,X_1)2^{-k\theta}\big)^p\Big)^{1/p}$$

(with a natural modification for $p = \infty$). For detailed descriptions of their properties, see [40].

Banach ideal spaces on $(0,\infty)$ can be taken as parameters of the real method of interpolation. In particular, it is easy to check [40, Lemma 3.1.3] that for any arbitrary $0 < \theta < 1$ and $1 \le p \le \infty$, we have

$$2^{-\theta}(\ln 2)^{1/p}\|x\|_{(X_0,X_1)_{\theta,p}} \le \Big(\int_0^\infty \big(t^{-\theta}\mathcal{K}(t,x;X_0,X_1)\big)^p \frac{dt}{t}\Big)^{1/p} \le 2(\ln 2)^{1/p}\|x\|_{(X_0,X_1)_{\theta,p}}.$$

In particular, the real method of interpolation is important because it provides all interpolation spaces for a quite wide class of Banach couples.

Definition 0.12. A Banach couple $\vec{X} = (X_0, X_1)$ is called *$\mathcal{K}$-monotone* (a *Calderon–Mityagin couple*) if from the inequality

$$\mathcal{K}(t,y;X_0,X_1) \;\le\; \mathcal{K}(t,x;X_0,X_1) \;\; (t>0)$$

with $x \in X_0 + X_1$ and $y \in X_0 + X_1$, it follows that an operator $U \in L(\vec{X},\vec{X})$ such that $y = Ux$ exists.

The second name is explained as follows: historically the first result in this direction is the theorem on the $\mathcal{K}$-monotonicity of the Banach couple (L_1, L_∞) proved by Calderon (see [53]) and Mityagin (see [108]). Using earlier expressions for the $\mathcal{K}$-functional of the Banach couple (L_1, L_∞), we provide an equivalent form of the Calderon–Mityagin theorem (see [92, Theorem 2.4.3]). Random variable X on $[0,1]$ is interpolation with respect to the couple (L_1, L_∞) if and only if the following condition is satisfied: if $x \in X$, $y \in L_1$, and

$$\int_0^t y^*(s)\,ds \le \int_0^t x^*(s)\,ds$$

for all $0 \le t \le 1$, then $y \in X$ and $\|y\|_X \le C\|x\|_X$, where the constant $C > 0$ depends only on the space X. In particular, it follows from this theorem that any maximal or separable symmetric space is interpolation with respect to the couple (L_1, L_∞).

Due to the Brudnyi–Kruglyak theorem, which is one of the main results in the theory of interpolation of operators (see [50, Theorem 15.1]), any interpolation space X with respect to a $\mathcal{K}$-monotone couple (X_0, X_1) can be presented as

$$X \;=\; (X_0, X_1)^{\mathcal{K}}_F,$$

where F is a parameter of the real method. In particular, the latter expression holds for a couple of weight L_p-spaces because any such couple is $\mathcal{K}$-monotone by virtue of the Sparr theorem (see [131]). This fact will be used below.

Detailed descriptions of results and facts from the theory of interpolation of operators given here (and many other ones) can be found in [39, 40, 50, 51, 91, 92, 102].

Chapter 1

ELEMENTARY PROPERTIES OF RADEMACHER FUNCTIONS

Definition 1.1. In the sequel, Δ_n^k are dyadic intervals from $[0,1]$, i.e.,

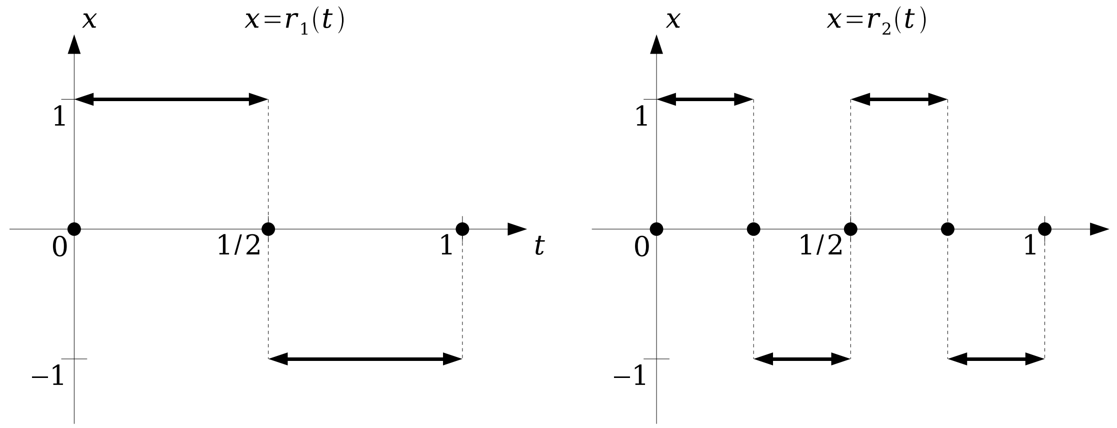
$$\Delta_n^k \;=\; \Big(\frac{k-1}{2^n}, \frac{k}{2^n}\Big) \;\; (n = 1,2,\dots; k = 1,2,\dots,2^n).$$

The Rademacher system $\{r_n\}_{n=1}^\infty$ consists of functions defined on $[0,1]$ as follows:

$$r_n(t) = (-1)^{k-1} \;\text{ if }\; t \in \Delta_n^k \; (k = 1,2,\dots,2^n) \;\text{ and }\; r_n(k2^{-n}) = 0 \; (k = 0,1,2,\dots,2^n).$$

A more compact (and equivalent) definition of the Rademacher functions can be given as follows:

$$r_n(t) = \operatorname{sign}\sin 2^n\pi t, \;\; 0 \le t \le 1.$$



First of all, we take into account the connection between functions $r_n(t)$ and the decomposition of numbers from the segment $[0,1]$ to binary fractions. Let a binary-irrational number $t \in [0,1]$ be defined as an infinite binary fraction

$$t = \sum_{i=1}^{\infty} \theta_i 2^{-i}, \quad \text{where} \quad \theta_i = \theta_i(t) = 0 \text{ or } 1.$$

Then for every $n = 1, 2, \dots$, we have

$$\sum_{i=1}^{n} \theta_i 2^{-i} < t < \sum_{i=1}^{n} \theta_i 2^{-i} + \sum_{i=n+1}^{\infty} 2^{-i} = \sum_{i=1}^{n} \theta_i 2^{-i} + 2^{-n},$$

and there exists a positive integer m such that

$$\frac{\theta_n + 2m}{2^n} < t < \frac{\theta_n + 2m + 1}{2^n}.$$

Therefore, it follows from the definition of Rademacher functions that $\theta_n(t) = 0$ only for binary-irrational t for such $r_n(t) = 1$. This yields that

$$r_n(t) = 1 - 2\theta_n(t) = (-1)^{\theta_n(t)}, \ \ t \in (0,1), \ t \neq \frac{j}{2^n} \ (n = 1, 2, \dots; j = 0, 1, \dots, 2^n). \tag{1.1}$$

Many important properties of the Rademacher system are due to its independence.

Proposition 1.1. *The Rademacher functions $r_n(t)$ $(n = 1, 2, \dots)$ are independent on $[0,1]$. For any $n \in \mathbb{N}$ and arbitrary functions $\varphi_k(x)$, $k = 1, 2, \dots, n$, we have*

$$\int_0^1 \varphi_1(r_1(t))\varphi_2(r_2(t)) \dots \varphi_n(r_n(t))\, dt = \prod_{k=1}^{n} \int_0^1 \varphi_k(r_1(t))\, dt. \tag{1.2}$$

Proof. Let $n \in \mathbb{N}$ and $\{I_k\}_{k=1}^{n}$ be an arbitrary set of intervals. Prove the following relation:

$$m\{x \in [0,1] : r_k(x) \in I_k, \ k = 1, 2, \dots, n\} = \prod_{k=1}^{n} m\{x \in [0,1] : r_k(x) \in I_k\}. \tag{1.3}$$

If there exists k such that $\{+1, -1\} \cap I_k = \varnothing$, then both parts of relation (1.3) are equal to 0. Otherwise it is reduced to

$$m\{x \in [0,1] : r_{k_i}(x) = \theta_i, \ i = 1, 2, \dots, m\} = \prod_{i=1}^{m} m\{x \in [0,1] : r_{k_i}(x) = \theta_i\}$$

where $1 \leq k_1 < k_2 < \ldots < k_m \leq n$ and $\theta_i = \pm 1$ $(i = 1, 2, \ldots, m)$. However, both parts of the last equation are equal to 2^{-m} by virtue of the definition of Rademacher functions. Let us check the equality (1.2). Since $\int\limits_0^1 \varphi_k(r_1(t))\,dt = (\varphi_k(1) + \varphi_k(-1))/2$ $(k = 1, 2, \ldots, n)$, it follows that the right-hand side of this relation is equal to

$$\prod_{k=1}^{n} \frac{1}{2}(\varphi_k(1) + \varphi_k(-1)) = \frac{1}{2^n} \sum_{\theta_k = \pm 1, k=1,\ldots,n} \varphi_1(\theta_1)\varphi_2(\theta_2)\ldots\varphi_n(\theta_n)$$
$$= \int\limits_0^1 \varphi_1(r_1(t))\varphi_2(r_2(t))\ldots\varphi_n(r_n(t))\,dt.$$

□

Corollary 1.1. *Let $\{A_j\}_{j=1}^m$ be an arbitrary set of disjoint subsets of natural series $\mathbb{N}$ and φ_j be a function depending on k_j real arguments, $k_j = \mathrm{card\,A_j}$, $j = 1, 2, \ldots, m$. Then functions $f_j(t) := \varphi_j(r_i(t), i \in A_j)$ form a system of independent functions.*

Corollary 1.2. *For any integer $k_1, k_2, \ldots, k_s$ and pairwise different $i_1, i_2, \ldots, i_s \in \mathbb{N}$, the integral*

$$\int\limits_0^1 r_{i_1}^{k_1}(t) r_{i_2}^{k_2}(t) \ldots r_{i_s}^{k_s}(t)\,dt$$

is equal to 1 *if all numbers $k_1, k_2, \ldots, k_s$ are even, and* 0 *if at least one of them is odd. In particular, $\{r_k\}$ is a strongly multiplicative system of functions on* $[0, 1]$.

Corollary 1.3. *The Rademacher functions form a orthonormal system.*

Remark 1.1. The Rademacher system is complete in $L_p[0, 1]$ for no $1 \leq p \leq \infty$. Indeed, according to Corollary 1.2, the function $z(t) := r_1(t)r_2(t)$ is orthogonal to all functions of this system, i.e., $\int\limits_0^1 z(t)r_k(t)\,dt = 0$ $(k = 1, 2, \ldots)$. This implies that $z(t)$ belongs to a closed linear envelope of the Rademacher system in $L_p[0, 1]$ for no $1 \leq p \leq \infty$.

Consider the question about convergence a.e. of Rademacher series. All necessary information related to this question is in the following two theorems.

Theorem 1.1. *If*

$$\sum_{k=1}^{\infty} a_k^2 < \infty, \tag{1.4}$$

then the series

$$\sum_{k=1}^{\infty} a_k r_k(t) \tag{1.5}$$

converges a.e. on $[0, 1]$.

We need the following auxiliary assertion.

Lemma 1.1. *Let a function $F(x)$ defined on $[0, 1]$ be differentiable at a point $t \in [0, 1]$. If $0 \leq a_k < t < b_k \leq 1$ and $b_k - a_k \to 0$ as $k \to \infty$, then*

$$\lim_{k\to\infty} \frac{F(b_k) - F(a_k)}{b_k - a_k} = F'(t).$$

Proof. Denoting $(b_k - t)/(b_k - a_k)$ by μ_k and $(t - a_k)/(b_k - a_k)$ by ν_k, we obtain the estimate

$$\Big|\frac{F(b_k) - F(a_k)}{b_k - a_k} - F'(t)\Big| = \Big|\frac{F(b_k) - F(t)}{b_k - t}\mu_k + \frac{F(t) - F(a_k)}{t - a_k}\nu_k - F'(t)\Big|$$
$$\leq \Big|\frac{F(b_k) - F(t)}{b_k - t} - F'(t)\Big|\mu_k + \Big|\frac{F(t) - F(a_k)}{t - a_k} - F'(t)\Big|\nu_k \leq \varepsilon(\mu_k + \nu_k) = \varepsilon$$

if k is sufficiently large. □

Proof of Theorem 1.1. Let $S_n = S_n(t)$ be a partial sum of series (1.5). Since $\{r_k\}$ is a orthonormal system on $[0, 1]$ by Corollary 1.3, we have

$$\|S_m - S_n\|_2 = \left(\sum_{k=n+1}^{m} a_k^2\right)^{1/2}$$

for arbitrary $1 \leq n < m$ according to Parseval's identity. Therefore, by Condition (1.4), due to the completeness of the space $L_2[0, 1]$, there exists $f \in L_2[0, 1]$ such that $\|S_n - f\|_2 \to 0$. Hence, for arbitrary $0 \leq a < b \leq 1$, we have

$$\lim_{n\to\infty} \int_a^b (f(t) - S_n(t))^2\, dt = 0.$$

This and the Cauchy–Bunyakovskii integral inequality yields:

$$\lim_{n\to\infty} \int_a^b (f(t) - S_n(t))\, dt = 0. \tag{1.6}$$

Let I_k be the interval of constancy of the function $r_k(t)$, and, therefore, of the function $S_k(t)$ $(k = 1, 2, \dots)$. One can see that

$$\int_{I_k} (S_n(t) - S_k(t))\, dt = 0$$

for $n > k$, where

$$\int_{I_k} f(t)\, dt = \int_{I_k} S_k(t)\, dt \tag{1.7}$$

due to (1.6).

Let E be a set of binary-irrational points of the segment $[0, 1]$ (i.e., points t, not represented as $p/2^k$, $p, k = 0, 1, 2, \dots$), where the function $F(t) := \int_0^t f(u)\, du$ is differentiable and $F'(t) = f(t)$. Since $f \in L_1[0, 1]$, it follows that $m(E) = 1$ (see [110, Theorem 9.4.2]). For any $t_0 \in E$ there exists a sequence of intervals $\{I_k\}_{k=1}^{\infty}$ such that $t_0 \in I_k$ $(k = 1, 2, \dots)$ and the function $S_k(t)$ is constant on I_k. Therefore, if $I_k = (a_k, b_k)$, then from (1.7) it follows that

$$S_k(t_0) = \frac{1}{b_k - a_k} \int_{I_k} S_k(t)\, dt = \frac{1}{b_k - a_k} \int_{I_k} f(t)\, dt.$$

Since $b_k - a_k \to 0$, it follows that the latter quantity converges to $F'(t_0) = f(t_0)$ for $k \to \infty$ by wirtue of Lemma 1.1. This means that (1.5) converges to any $t_0 \in E$, i.e., a.e. on $[0, 1]$. □

Theorem 1.2. *If Condition* (1.4) *does not hold, i.e.,* $\sum_{k=1}^{\infty} a_k^2 = \infty$, *then* (1.5) *diverges a.e. on* $[0, 1]$.

Proof. Suppose that the assumption of the theorem does not hold and the series (1.5) converges on a set $B \subset [0,1]$ of a positive measure. Then if $S_n(t)$ are partial sums of (1.5), then this set can be represented as the union of increasing sets B_k, where B_k are sets of $t \in B$ such that

$$|S_{n+p}(t) - S_n(t)| \le 1 \tag{1.8}$$

for all $n \ge k$ and $p \in \mathbb{N}$. Due to continuity of the Lebesgue measure, there exists k_0 such that $m(B_{k_0}) > 0$. Squaring and integrating (1.8) for $k = k_0$, we obtain the inequality

$$\int\limits_{B_{k_0}} (S_{n+p}(t) - S_n(t))^2\, dt \le m(B_{k_0}).$$

For $n \ge k_0$, this yields

$$\begin{aligned} m(B_{k_0}) \ge \int\limits_{B_{k_0}} \left(\sum_{i=n+1}^{n+p} a_i r_i(t)\right)^2 dt &= m(B_{k_0}) \sum_{i=n+1}^{n+p} a_i^2 \\ &+ 2 \sum_{n+1 \le i < j \le n+p} a_i a_j \int\limits_{B_{k_0}} r_i(t) r_j(t)\, dt. \end{aligned} \tag{1.9}$$

By virtue of Corollary 1.2, functions $r_i(t)r_j(t)$ $(1 \le i < j < \infty)$ form a orthonormal system on $[0,1]$. Therefore, due to the Bessel inequality, we have

$$\sum_{n+1 \le i < j \le n+p} \left(\int\limits_{B_{k_0}} r_i(t) r_j(t)\, dt \right)^2 \le \frac{(m(B_{k_0}))^2}{9}$$

for $n \ge k_1 \ge k_0$ (the left-hand side is the sum of squares of the Fourier coefficients for the function $\chi_{B_{k_0}}$ with respect to the system $\{r_i(t)r_j(t)\}_{1 \le i < j < \infty}$). This and the Cauchy–Bunyakovskii inequality imply that the absolute value of the second summand on the right-hand side in (1.9) is less that

$$\frac{m(B_{k_0})}{3} \left(\sum_{i,j} (a_i a_j)^2 \right)^{1/2} \le \frac{m(B_{k_0})}{3} \sum_{i=n+1}^{n+p} a_i^2.$$

Therefore, from (1.9), we obtain the inequality

$$m(B_{k_0}) \ge \frac{m(B_{k_0})}{3} \sum_{i=n+1}^{n+p} a_i^2,$$

i.e., $\sum\limits_{i=n+1}^{n+p} a_i^2 \le 3$ for $n \ge k_1$ and arbitrary $p \in \mathbb{N}$. Thus (1.4) holds, but this contradicts the assumption of the theorem. □

The following classical result became the source of many investigations and will play a very important role.

Theorem 1.3. *Let $p > 0$ and $S_n(t) = \sum\limits_{i=1}^{n} a_i r_i(t)$ be an arbitrary polynomial on the Rademacher system with real coefficients $a_1, a_2, \dots, a_n$. Then*

$$\int\limits_0^1 |S_n(t)|^p\, dt \le \left(\frac{p}{2} + 1\right)^{p/2} \left(\sum_{i=1}^{n} a_i^2\right)^{p/2}. \tag{1.10}$$

Proof. First, consider the case for $p = 2m$, $m \geq 1$. Due to the polynomial theorem and Corollary 1.2, we have

$$\int_0^1 |S_n(t)|^{2m}\, dt = \sum_{k_1+\dots+k_n=m, k_i\geq 0} \frac{(2m)!}{(2k_1)!\dots(2k_n)!}|a_1|^{2k_1}\dots|a_n|^{2k_n}$$

where summation is over all integer $k_i \geq 0$ such that $k_1 + \dots + k_n = m$.

Since $2^{k_i} k_i! \leq (2k_i)!$ $(i = 1, 2, \dots, n)$, it follows that $2^m\, k_1! \dots k_n! \leq (2k_1)! \dots (2k_n)!$ Therefore, due to the polynomial theorem, we have the inequality

$$\int_0^1 |S_n(t)|^{2m}\, dt \leq \frac{(2m)!}{2^m m!} \sum_{k_1+\dots+k_n=m, k_i\geq 0} \frac{m!}{k_1!\dots k_n!}|a_1|^{2k_1}\dots|a_n|^{2k_n} = \frac{(2m)!}{2^m m!}\left(\sum_{i=1}^n a_i^2\right)^m; \tag{1.11}$$

thus, inequality (1.10) is proved for $B_{2m} := (2m)!/(2^m m!)$. Here $B_{2m} \leq m^m$ because $(m+1)(m+2)\dots 2m \leq (2m)^m$.

Consider the case of an arbitrary $p > 0$. Then there exists $m \in \mathbb{N}$ such that $2m - 2 < p \leq 2m$. Since $\|f\|_p$ increases with respect to p, using (1.11), we have

$$\|S_n\|_p = \left(\int_0^1 |S_n(t)|^p\, dt\right)^{1/p} \leq \left(\int_0^1 |S_n(t)|^{2m}\, dt\right)^{1/(2m)}$$
$$\leq m^{1/2}\left(\sum_{i=1}^n a_i^2\right)^{1/2} \leq \left(\frac{p}{2}+1\right)^{1/2}\left(\sum_{i=1}^n a_i^2\right)^{1/2}.$$

Thus, (1.10) is proved. □

Remark 1.2. Let ξ be a standard Gaussian random variable, i.e.,

$$\mathbb{P}\{\xi < x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt.$$

Then its moment $\mathbb{E}\xi^{2m}$ of order $2m$ coincides with the constant B_{2m} from the last proof (see [117, Sec. 2, p. 11]). Moreover, the same arguments as in the proof of Theorem 1.3 show that the right-hand side of (1.11) is equal to $\mathbb{E}\Big(\sum\limits_{k=1}^n a_k\xi_k\Big)^{2m}$, where $\xi_1, \dots, \xi_n$ are independent standard normal random variables. Indeed, due to the mentioned property of ξ and the independence of ξ_k, we have

$$\begin{aligned}
\mathbb{E}\Big(\sum_{k=1}^n a_k\xi_k\Big)^{2m} &= \sum_{k_1+\dots k_n=m, k_i\geq 0} \frac{(2m)!}{(2k_1)!\dots(2k_n)!}|a_1|^{2k_1}\dots|a_n|^{2k_n}\mathbb{E}\xi_1^{2k_1}\dots\mathbb{E}\xi_n^{2k_n} \\
&= \sum_{k_1+\dots k_n=m, k_i\geq 0} \frac{(2m)!}{(2k_1)!\dots(2k_n)!}\cdot\frac{(2k_1)!\dots(2k_n)!}{2^{k_1}\dots 2^{k_n}(k_1)!\dots(k_n)!}|a_1|^{2k_1}\dots|a_n|^{2k_n} \\
&= \frac{(2m)!}{2^m m!}\sum_{k_1+\dots+k_n=m, k_i\geq 0}\frac{m!}{k_1!\dots k_n!}|a_1|^{2k_1}\dots|a_n|^{2k_n} = \frac{(2m)!}{2^m m!}\left(\sum_{i=1}^n a_i^2\right)^m.
\end{aligned}$$

Therefore, inequality (1.11) can be presented as a comparison of even moments of the polynomials with respect to the systems $\{r_k\}$ and $\{\xi_k\}$:

$$\mathbb{E}\Big(\sum_{k=1}^n a_k r_k\Big)^{2m} = \int_0^1 \Big|\sum_{k=1}^n a_k r_k(t)\Big|^{2m}\, dt \leq \mathbb{E}\Big(\sum_{k=1}^n a_k\xi_k\Big)^{2m}. \tag{1.12}$$

Corollary 1.4. *If* $\sum_{k=1}^{\infty} a_k^2 < \infty$, *then series* $\sum_{k=1}^{\infty} a_k r_k(t)$ *converges in space* L_p *for any* $p > 0$.

Proof. Let $m > n \geq 1$. Substituting the difference $S_m - S_n$ for S_n in (1.10), one can check that the sequence of partial sums of series $\sum_{k=1}^{\infty} a_k r_k(t)$ is a Cauchy sequence in L_p. Thus, due to the completeness of this space, the corollary is proved. □

Corollary 1.5. *There exists a universal constant* $C > 0$ *such that*

$$m\{t \in [0,1] : |S_n(t)| > u\} \leq C \exp\left(-\frac{u^2}{2\sum_{k=1}^{n} a_k^2}\right) \quad (u > 0) \tag{1.13}$$

for any polynomial $S_n(t) = \sum_{i=1}^{n} a_i r_i(t)$.

Proof. Applying the Stirling formula (see [101, p. 31])

$$n! = \sqrt{2\pi n} n^n e^{-n} e^{R_n} \quad \text{for} \quad \frac{1}{12n+1} < R_n < \frac{1}{12n}, \tag{1.14}$$

we estimate the constant B_{2m} from the proof of Theorem 1.3 as follows:

$$B_{2m} \leq \sqrt{2} 2^m e^{-m} m^m.$$

Therefore, from the Chebychev inequality and relation (1.11) for $m = \left[\frac{u^2}{2\sum_{k=1}^{n} a_k^2}\right]$, it follows that

$$m\{t \in [0,1] : |S_n(t)| > u\} \leq u^{-2m} \mathbb{E}|S_n|^{2m} \leq B_{2m} u^{-2m} \Big(\sum_{k=1}^{n} a_k^2\Big)^m$$

$$\leq \sqrt{2} \left(\frac{2m \sum_{k=1}^{n} a_k^2}{eu^2}\right)^m \leq \sqrt{2} e^{-m} \leq e\sqrt{2} \exp\left(-\frac{u^2}{2\sum_{k=1}^{n} a_k^2}\right).$$

□

Remark 1.3. It is easy to get the inverse assertion: inequality (1.10) follows from (1.13). To do that, it suffices to use the well-known norm relation for functions in $L_p[0,1]$ (see [86, Appendix 1, Sec. 1, Proposition 1]):

$$\|f\|_p^p = p \int_0^{\infty} u^{p-1} m\{t \in [0,1] : |f(t)| > u\}\, du.$$

Using Theorem 1.3, one can establish the following fundamental fact of polynomial norm equivalence on the Rademacher system in $L_p[0,1]$ for all $1 \leq p < \infty$.

Theorem 1.4 (the Khintchine inequality)**.** *For any* $1 \leq p < \infty$ *there exists a constant* $C_p > 0$ *such that for all* $n \in \mathbb{N}$ *and* $a = (a_k)_{k=1}^n$ *the following inequality holds:*

$$\frac{1}{2}\|a\|_2 \leq \Big\|\sum_{k=1}^{n} a_k r_k\Big\|_p \leq C_p \|a\|_2. \tag{1.15}$$

Proof. The validity of the right-hand side of (1.15), which is the most important inequality, is proved in Theorem 1.3 with $C_p = (p/2+1)^{1/2}$. Prove the left-hand side.

Due to Parseval's identity and the Hölder inequality with $q = 3/2$ and $q' = 3$, we have

$$\|a\|_2^2 = \mathbb{E}S_n^2 = \left(\int_0^1 |S_n(t)|^{2/3}|S_n(t)|^{4/3}\,dt\right) \le \left(\int_0^1 |S_n(t)|\,dt\right)^{2/3} \left(\int_0^1 (S_n(t))^4\,dt\right)^{1/3},$$

where $S_n = \sum_{k=1}^{n} a_k r_k$. Then, applying (1.11) for $m = 2$, we obtain the inequality

$$\|a\|_2^2 \le \|S_n\|_1^{2/3}(4\|a\|_2^4)^{1/3},$$

whence $\|a\|_2 \le 2\|S_n\|_1$. Inequality (1.15) is proved because $\|f\|_1 \le \|f\|_p$ for $p \ge 1$,. □

Corollary 1.6. *For any set of functions $\{f_i\}_{i=1}^n \subset L_p = L_p(T,\Sigma,\mu)$, where (T,Σ,μ) is a space with σ-finite measure μ, the following inequality holds:*

$$\frac{1}{2}\left\|\left(\sum_{i=1}^n f_i^2\right)^{1/2}\right\|_p \le \left\{\int_0^1 \left\|\sum_{i=1}^n f_i r_i(s)\right\|_p^p ds\right\}^{1/p} \le C_p \left\|\left(\sum_{i=1}^n f_i^2\right)^{1/2}\right\|_p. \tag{1.16}$$

Proof. By Theorem 1.4, for every fixed $t \in T$, we have

$$\frac{1}{2}\left(\sum_{i=1}^n f_i^2(t)\right)^{1/2} \le \left\{\int_0^1 \left|\sum_{i=1}^n f_i(t) r_i(s)\right|^p ds\right\}^{1/p} \le C_p \left(\sum_{i=1}^n f_i^2(t)\right)^{1/2}.$$

Since

$$\int_T \int_0^1 \left|\sum_{i=1}^n f_i(t) r_i(s)\right|^p ds\,d\mu = \int_0^1 \int_T \left|\sum_{i=1}^n f_i(t) r_i(s)\right|^p d\mu\,ds$$

by Fubini's theorem, inequality (1.16) is a corollary of the previous inequality due to the monotonicity of the norm in L_p. □

Remark 1.4. On the left-hand side of inequality (1.15), the factor $1/2$ can be replaced by $1/\sqrt{2}$ and, as was proved in [134], this constant is the best for $p = 1$, i.e., this is the greatest possible value such that the following inequality holds for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$:

$$c\Big(\sum_{k=1}^n a_k^2\Big)^{1/2} \le \Big\|\sum_{k=1}^n a_k r_k\Big\|_1.$$

The least possible value of C_p from the right-hand side of inequality (1.15) for all p was found in [73]; note that $\lim\limits_{p\to\infty} C_p\sqrt{e/p} = 1$.

Remark 1.5. Inequality (1.15) is not valid for $p = \infty$. Indeed, due to the definition of Rademacher functions for any $n \in \mathbb{N}$ and an arbitrary sign setting $\theta_k = \pm 1$ $(k = 1, 2, \dots, n)$, there exists $i = 1, 2, \dots, 2^n$ such that $\theta_k = r_k(t)$ $(k = 1, 2, \dots, n)$ for all $t \in \Delta_n^i$. Therefore, for any real $a_1, \dots, a_n$, the following equality holds:

$$\Big\|\sum_{k=1}^n a_k r_k\Big\|_\infty = \sum_{k=1}^n |a_k|.$$

The boundedness of the sum of Rademacher series on an interval guarantees the absolute convergence of the series of its coefficients. This assertion complements the previous remark.

Proposition 1.2. *Suppose that Condition* (1.4) *holds. Then, if*

$$\sum_{k=1}^{\infty} a_k r_k \in L_\infty(\Delta) \tag{1.17}$$

for any (*nonempty*) *interval* $\Delta \subset [0,1]$, *then* $\sum\limits_{k=1}^{\infty} |a_k| < \infty$.

Proof. It is clear that we can take a dyadic interval Δ_m^j $(m = 1, 2, \ldots; j = 1, 2, \ldots, 2^m)$ as Δ in (1.17). Then since functions $\sum\limits_{k=m+1}^{\infty} a_k r_k \cdot \chi_{\Delta_m^i}$ $(i = 1, 2, \ldots, 2^m)$ are equimeasurable, we have

$$\Big|\sum_{k=1}^{\infty} a_k r_k(t)\Big| \le \sum_{k=1}^{m} |a_k| + \Big|\sum_{k=m+1}^{\infty} a_k r_k(t)\Big| \le C$$

for all $t \in [0,1]$, where

$$C = 2\sum_{k=1}^{m} |a_k| + \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{L_\infty(\Delta)}.$$

Fix an arbitrary positive integer n and consider a dyadic interval Δ_n^k $(k = 1, 2, \ldots, 2^n)$ such that $r_i(t) = \operatorname{sign} a_i$ $(i = 1, 2, \ldots, n)$ if $t \in \Delta_n^k$. Since $\int\limits_{\Delta_n^k} r_i(t)\,dt = 0$ for $i > n$, it follows that

$$\sum_{i=1}^{n} |a_i| 2^{-n} = \sum_{i=1}^{n} a_i \int\limits_{\Delta_n^k} r_i(t)\,dt = \int\limits_{\Delta_n^k} \sum_{i=1}^{\infty} a_i r_i(t)\,dt \le C2^{-n}.$$

So $\sum\limits_{i=1}^{n} |a_i| \le C$ for any $n \in \mathbb{N}$; the statement is proved. □

In Chap. 6, we need the following version of the Khintchine inequality for the orthonormal system $\{r_i r_j\}_{1 \le i < j < \infty}$ on $[0,1]$.

Proposition 1.3. *There exist absolute constants* $A > 0$ *and* $B > 0$ *such that for any* $n \in \mathbb{N}$ *and* $c_{i,j} \in \mathbb{R}$ $(1 \le i < j \le n)$, *the following inequalities hold:*

$$\int\limits_0^1 \Big(\sum_{1 \le i < j \le n} c_{i,j} r_i(t) r_j(t)\Big)^4 dt \le A\left(\sum_{1 \le i < j \le n} c_{i,j}^2\right)^2 \tag{1.18}$$

and

$$\left(\sum_{1 \le i < j \le n} c_{i,j}^2\right)^{1/2} \le B \int\limits_0^1 \Big|\sum_{1 \le i < j \le n} c_{i,j} r_i(t) r_j(t)\Big|\,dt. \tag{1.19}$$

Proof. Applying Corollary 1.2 (in the same way as in the proof of Theorems 1.3 and 1.4) we obtain

$$\int\limits_0^1 \left(\sum_{1 \le i < j \le n} c_{i,j} r_i(t) r_j(t)\right)^4 dt = \sum_{1 \le i < j \le n} c_{i,j}^4 + 6 \sum_{\substack{1 \le i < j \le n,\\ 1 \le k < m \le n,\\ i \ne k \text{ or } j \ne m}} c_{i,j}^2 c_{k,m}^2$$

$$+ 24 \sum_{1 \le i < k < j < m \le n} (c_{i,j} c_{k,m} c_{i,k} c_{j,m} + c_{i,j} c_{k,m} c_{i,m} c_{k,j} + c_{i,k} c_{j,m} c_{i,m} c_{k,j}).$$

Since due to the Cauchy–Bunyakovskii inequality, the last sum does not exceed $A\Big(\sum\limits_{1\le i<j\le n} c_{i,j}^2\Big)^2$, inequality (1.18) is proved.

Denote $T_n = \sum\limits_{1\le i<j\le n} c_{i,j} r_i r_j$. Arguing in the same way as in the second part of the proof of Theorem 1.4 and applying (1.18), we obtain the inequality $\|(c_{i,j})\|_2^2 \le A^{1/3}\|T_n\|_1^{2/3}\|(c_{i,j})\|_2^{4/3}$. Inequality (1.19) follows directly from the last inequality. □

The following assertion concerns the comparison of L_p- and L_q-norms of the Rademacher polynomials. It makes the Khintchine inequality more exact.

Theorem 1.5. *For any $1 < p \le q < \infty$, $n \in \mathbb{N}$, and $a_1, a_2, \dots, a_n \in \mathbb{R}$, the following inequality holds:*

$$\Big\|\sum_{i=1}^n a_i r_i\Big\|_q \le \sqrt{\frac{q-1}{p-1}}\Big\|\sum_{i=1}^n a_i r_i\Big\|_p. \tag{1.20}$$

To prove it, we need one numerical inequality.

Lemma 1.2. *Let $1 < p \le q < \infty$ and $l = \sqrt{(p-1)/(q-1)}$. Then, for arbitrary $u \ge 0$ and $v \ge 0$, the following inequality holds:*

$$\left(\frac{|v+lu|^q + |v-lu|^q}{2}\right)^{1/q} \le \left(\frac{|v+u|^p + |v-u|^p}{2}\right)^{1/p}.$$

Proof. Due to the homogeneity, it suffices to consider one case $v = 1$ (if $v = 0$, then the inequality is obvious). First, assume that $1 < p \le q \le 2$. Then the binomial coefficients

$$C_{2k}^q = \frac{q(q-1)\dots(q-2k+1)}{(2k)!} \quad \text{and} \quad C_{2k}^p = \frac{p(p-1)\dots(p-2k+1)}{(2k)!}$$

are non-negative and one can check that $C_{2k}^q l^2 \le qp^{-1}C_{2k}^p$. Therefore, using the power series expansion $f(x) = (1+x)^r$ and the simple inequality $(1+x)^r \le 1 + rx$ $(x \ge 0, 0 \le r \le 1)$, we obtain

$$\begin{aligned}\left(\frac{(1+lu)^q + (1-lu)^q}{2}\right)^{p/q} &= \left(1 + \sum_{k=1}^\infty C_{2k}^q l^{2k} u^{2k}\right)^{p/q} \\ &\le 1 + \sum_{k=1}^\infty C_{2k}^p u^{2k} = \frac{(1+u)^p + (1-u)^p}{2}.\end{aligned}$$

Therefore, the lemma is proved for $u \le 1$. If $u > 1$, then it suffices to apply the inequality $|1 \pm lu| \le |u \pm l|$ $(|l| \le 1,\ |u| \ge 1)$, which can be easily checked, and the fact that the proving inequality is proved for $1/u$:

$$\begin{aligned}\left(\frac{|1+lu|^q + |1-lu|^q}{2}\right)^{1/q} &\le \left(\frac{|u+l|^q + |u-l|^q}{2}\right)^{1/q} \\ \le\ u\left(\frac{|1+1/u|^p + |1-1/u|^p}{2}\right)^{1/p} &= \left(\frac{|1+u|^p + |1-u|^p}{2}\right)^{1/p}.\end{aligned}$$

Thus, the case $1 < p \le q \le 2$ is completed.

Further, define the map

$$Tf(s) = \int_0^1 f(t)\,dt + l\int_0^1 f(t) r_1(t)\,dt \cdot r_1(s).$$

Note that the inequality that we have proved holds if and only if the norm of T treated as an operator from $L_p[0,1]$ to $L_q[0,1]$ is not more than 1. Indeed, one can check that an analogous statement is true for the restriction of the operator T to the two-dimensional space $M := \{a + br_1(s),\ a, b \in \mathbb{R}\}$. At the same time, for any function $f \in L_p$, we have $Tf = TPf$, where P is the orthogonal projector in L_p on M, i.e.,

$$Pf(s) = \int_0^1 f(t)\,dt + \int_0^1 f(t)r_1(t)\,dt \cdot r_1(s).$$

Since calculations show that the norm P in L_p is equal to 1, it follows that the norm T from L_p to L_q coincides with the respective norm of the restriction of T to M. Then the statement holds.

Moreover, one can easily check that the operator T is self-adjoint, i.e., $T^* = T$, and $l = \sqrt{(q'-1)/(p'-1)}$. Therefore, the norm of this operator from $L_p[0,1]$ to $L_q[0,1]$ is not greater then 1 if $2 \le p \le q < \infty$. Therefore, the proving inequality holds in this case. Since $1 < p < 2 < q < \infty$, it suffices to note that $l = l_1 l_2$, where $l_1 = \sqrt{p-1}$ and $l_2 = 1/\sqrt{q-1}$. This implies that

$$\left(\frac{|v+lu|^q + |v-lu|^q}{2}\right)^{1/q} \le \left(\frac{|v+l_1u|^2 + |v-l_1u|^2}{2}\right)^{1/2} \le \left(\frac{|v+u|^p + |v-u|^p}{2}\right)^{1/p}.$$

$\square$

Proof of Theorem 1.5. Let $l = \sqrt{(p-1)/(q-1)}$ $(1 < p \le q < \infty)$. It suffices to prove that

$$\left(\int_0^1 \left|a_0 + l\sum_{i=1}^n a_i r_i(t)\right|^q dt\right)^{1/q} \le \left(\int_0^1 \left|a_0 + \sum_{i=1}^n a_i r_i(t)\right|^p dt\right)^{1/p} \tag{1.21}$$

for any $a_0, a_1, \dots, a_n \in \mathbb{R}$.

Apply the method of mathematical induction. First of all, for $n = 1$, inequality (1.21) coincides with the just proved inequality from Lemma 1.2. Suppose that (1.21) holds for n and consider an arbitrary set of numbers $\{a_i\}_{i=0}^{n+1}$. By the induction hypothesis, we have

$$R_{n+1} := \left(\int_0^1 \left|a_0 + l\sum_{i=1}^{n+1} a_i r_i(t)\right|^q dt\right)^{1/q}$$

$$= \left(\frac{1}{2}\int_0^1 \left(\left|a_0 + la_{n+1} + l\sum_{i=1}^n a_i r_i(t)\right|^q + \left|a_0 - la_{n+1} + l\sum_{i=1}^n a_i r_i(t)\right|^q\right) dt\right)^{1/q}$$

$$\le 2^{-1/q}(\|F\|_p^q + \|G\|_p^q)^{1/q},$$

where

$$F(t) := a_0 + la_{n+1} + \sum_{i=1}^n a_i r_i(t) \ \text{ and } \ G(t) := a_0 - la_{n+1} + \sum_{i=1}^n a_i r_i(t).$$

Note that the following inequality holds:

$$(\|F\|_p^q + \|G\|_p^q)^{1/q} \le \|(|F(t)|^q + |G(t)|^q)^{1/q}\|_p.$$

Indeed, if $\alpha = p/q$, then it is easy to see that it is equivalent to the Minkowski inequality $\|f\|_\alpha + \|g\|_\alpha \le \|f+g\|_\alpha$ for the L_α-norm $(0 < \alpha < 1)$. Therefore, from the previous inequality and Lemma 1.2, for $v = a_0 + \sum\limits_{i=1}^n a_i r_i(t)$ and $u = a_{n+1}$, we have

$$R_{n+1} \le 2^{-1/q}\|(|F(t)|^q + |G(t)|^q)^{1/q}\|_p \le \left(\int_0^1 \left|a_0 + \sum_{i=1}^{n+1} a_i r_i(t)\right|^p dt\right)^{1/p}.$$

$\square$

In particular, Theorem 1.4 and Remark 1.5 show that the Rademacher system is a symmetric basic sequence in the space $L_p[0,1]$ for any $1 \le p \le \infty$. Prove that this property of the Rademacher system holds for any symmetric space

Proposition 1.4. *If $(a_k)_{k=1}^{\infty} \in \ell_2$ and $\pi = \pi(k)$ is an arbitrary permutation of positive integers, then the absolute values of the functions*

$$x(t) = \sum_{k=1}^{\infty} a_k r_k(t) \quad \text{and} \quad y(t) = \sum_{k=1}^{\infty} |a_{\pi(k)}| r_k(t)$$

are equimeasurable.

Proof. First of all, by virtue of Theorem 1.1, $x(t)$ and $y(t)$ are measurable and finite a.e. on $[0,1]$. Treat them as random variables on this segment, we find the characteristic functions of the respective distributions. By virtue of Lemma 1.1, we have

$$\varphi_x(t) = \int_0^1 \exp(itx(u))\, du = \prod_{k=1}^{\infty} \int_0^1 \exp(ita_k r_k(u))\, du = \prod_{k=1}^{\infty} \cos(t|a_k|). \tag{1.22}$$

Similarly, we have

$$\varphi_y(t) = \prod_{k=1}^{\infty} \cos(t|a_{\pi(k)}|). \tag{1.23}$$

Since $\cos z = 1 - 2\sin^2(z/2)$ and the series $\sum\limits_{k=1}^{\infty} \sin^2(t|a_k|/2)$ converges by assumption, it follows that product (1.22) absolutely converges for any $t \in \mathbb{R}$. Hence, product (1.22) coincides with (1.23) for any $t \in \mathbb{R}$, i.e., $\varphi_x(t) = \varphi_y(t)$ $(t \in \mathbb{R})$. By the uniqueness theorem (see [46, corollary of Theorem 6.3.2, p. 141]), any characteristic function uniquely defines the distribution function of the random quantity, i.e.,

$$m\{t \in [0,1] : x(t) < \tau\} = m\{t \in [0,1] : y(t) < \tau\} \quad (\tau \in \mathbb{R});$$

in particular, this implies the necessary statement. □

Corollary 1.7. *The Rademacher system is a symmetric basic sequence in any symmetric space X on $[0,1]$. Moreover, if $(a_k^*)_{k=1}^{\infty}$ is a decreasing permutation of the sequence $(|a_k|)_{k=1}^{\infty}$, where $(a_k)_{k=1}^{\infty} \in \ell_2$, then the functions $\sum\limits_{k=1}^{\infty} a_k r_k$ and $\sum\limits_{k=1}^{\infty} a_k^* r_k$ belong or not to the space X simultaneously and the following relation holds:*

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X = \Big\| \sum_{k=1}^{\infty} a_k^* r_k \Big\|_X. \tag{1.24}$$

Proof. Let us check that $\{r_k\}_{k=1}^{\infty}$ is a monotone basic sequence in symmetric space X. In other words, show that for any $n \in \mathbb{N}$ and any $a_k \in \mathbb{R}$ $(k = 1, 2, \dots, n+1)$ the following inequality holds:

$$\Big\| \sum_{k=1}^{n} a_k r_k \Big\|_X \le \Big\| \sum_{k=1}^{n+1} a_k r_k \Big\|_X. \tag{1.25}$$

Indeed, consider the transformation ω mapping the segment $[0,1]$ onto itself and permuting neighboring dyadic intervals Δ_{n+1}^{2m-1} and Δ_{n+1}^{2m} $(m = 1, 2, \dots, 2^n)$, where the function r_{n+1} takes values $+1$ and -1 respectively. It is obvious that ω preserves measure and, therefore, by the definition of a symmetric space, if $y := \sum\limits_{k=1}^{n+1} a_k r_k$, then $\|y(\omega)\|_X = \|y\|_X$. Then, applying the triangle inequality to the equality $\sum\limits_{k=1}^{n} a_k r_k = (y + y(\omega))/2$, we have (1.25).

Then since $(a_k)_{k=1}^\infty \in \ell_2$, it follows that there exists a permutation π of the set of integers such that $a_k^* = |a_{\pi(k)}|$. Therefore, relation (1.24) and the symmetry of $\{r_k\}_{k=1}^\infty$ in X follow from Proposition 1.4 and the definition of symmetric space □

Finally we prove an assertion concerning the comparison of Rademacher sums with respect to decreasing step functions. We need it in Chaps. 4 and 5.

Proposition 1.5. *If*

$$x(t) = \sum_{k=1}^\infty a_k r_k(t), \quad a = (a_k)_{k=1}^\infty \in \ell_2,$$

then

$$\sum_{k=1}^\infty a_k^* \chi_{(0,2^{-k-1})}(t) \le x^*(t).$$

Proof. Introduce the functions

$$y(t) = \sum_{k=1}^\infty a_k^* r_k(t) \quad \text{and} \quad y_m(t) = \sum_{k=1}^m a_k^* r_k(t) \ (m \in \mathbb{N}),$$

where $(a_k^*)_{k=1}^\infty$ is a decreasing permutation of the sequence $(|a_k|)_{k=1}^\infty$. For arbitrary $l < m$, the latter function reads as

$$y_m(t) = a_1^* r_1(t) + a_2^* r_2(t) + \dots + a_l^* r_l(t) + \sum_{k=l+1}^m a_k^* r_k(t).$$

Since the last summand considered on $[0, 2^{-l}]$ is symmetrically distributed, it follows that there exists the set $E_l \subset [0, 2^{-l}]$, $m(E_l) \ge 2^{-l-1}$, where this summand is non-negative. Therefore, $y_m(t) \ge a_1^* + a_2^* \dots + a_l^*$ for $t \in E_l$. Hence, for all $l \le m$, we have

$$y_m^*(t) \ge \sum_{k=1}^l a_k^* \chi_{(0,2^{-k-1})}(t) \ \ (0 \le t \le 1).$$

By Theorem 1.1, $y_m(t) \to y(t)$ as $m \to \infty$ for almost all $t \in [0,1]$. Therefore, $y_m^*(t) \to y^*(t)$ a.e. on $[0,1]$ (see [92, Chap. 2, Sec. 2, property 11°, p. 93]). Since $y^*(t) = x^*(t)$ due to Proposition 1.4, it follows from the previous inequality that

$$\sum_{k=1}^l a_k^* \chi_{(0,2^{-k-1})}(t) \le x^*(t)$$

for all $l \in \mathbb{N}$ and $t \in [0,1]$. Finally, passing to the limit as $l \to \infty$, we obtain the required inequality. □

Corollary 1.8. *Let X be a symmetric space on $[0,1]$, $a = (a_k)_{k=1}^\infty \in \ell_2$. If*

$$x_a(t) = \sum_{k=1}^\infty a_k r_k(t) \quad \text{and} \quad f_a(t) = \sum_{k=1}^\infty a_k^* \chi_{(0,2^{-k+1})}(t),$$

then $\|f_a\|_X \le 4\|x_a\|_X$.

Proof. It suffices to note that

$$f_a(t) = \sigma_4\Big(\sum_{k=1}^\infty a_k^* \chi_{(0,2^{-k-1})}(t)\Big),$$

where $\sigma_\tau x(t) = x(t/\tau)$, and use the fact that $\|\sigma_\tau\|_{X\to X} \le \max(1,\tau)$ (see [92, Sec. 2.4, Corollary 1]). □

Comments and References. Rademacher functions $r_n(x)$ $(n = 1, 2, \dots)$ were first introduced in 1922 by Rademacher (see [121]), using the expansion of a number $t \in [0,1]$ into a binary fraction (see relations (1.1)). Let us note that in probability theory this system was known long before as the sequence of independent equally and symmetrically distributed random values taking the value ± 1. The behavior of sums $\sum\limits_{k=1}^{n} r_k(x)$ as $n \to \infty$ was investigated in connection with the strong Borel law of large numbers and its refinings (see [117, Sec. 1]).

There exists another frequent definition of the Rademacher functions as characters on a compact Cantor Abelian group $\mathcal{K} = \{-1,1\}^{\mathbb{N}}$ of functions mapping $\mathbb{N}$ to the two-point set $\{-1,1\}$ (see, e.g., [72, Sec. 1.2], [66, Sec. 14.1], and [42]). The group $\mathcal{K}$ is supplied with the discrete topology and natural product and the nth Rademacher character r_n is defined as follows: $r_n(\omega) = \omega(n)$ $(\omega \in \mathcal{K})$.

Note that the Rademacher system is a particular case of the following class of functional sequences. Let function ψ from $L_2(-\infty,\infty)$ be 1-periodic and $\psi(t) + \psi(t + 1/2) = 0$ for all $t \in \mathbb{R}$. Assume that $\psi_n(t) = \psi(2^{n-1}t)$ $(n = 1, 2, \dots)$. One can easily check that $\{\psi_n\}$ is an orthogonal system of functions on $[0,1]$ (see [81, Sec. 2.2, p. 56]). In particular, if $\psi(t) = \operatorname{sign} \sin 2\pi t$, then we obtain the Rademacher system.

Chapter 1 contains mostly standard and well-known facts about the Rademacher functions. The system $\{r_n\}_{n=1}^{\infty}$ is a simple and very important example of an independent function sequence. The notion of independence is central in probability theory. It is useful in function theory as well. This method of construction for sequences of independent functions was applied in the 30s of the previous century in the monographs of Polish mathematicians (Steinhaus, Banach, Marcinkiewicz, Zygmund, and others). It is given in Secs. 2.1 and 2.2 of the monograph [86]. In particular, (1.2) holds for any sets of independent functions (see [86, Theorem 2.2.4]).

Theorem 1.1 was proved by Rademacher (see [121]). Theorem 1.2, which is the unverse result in a way, was proved by Kolmogorov (see [89]) (in fact, a more general result was proved for systems of independent functions under certain conditions). The definitions of these theorems were taken from [81, Sec. 4.5].

Inequality (1.10) was proved in [87] to get "good" measure estimates for

$$m\Big\{t \in [0,1] : \Big|\sum_{k=1}^{n} a_k r_k(t)\Big| \geq \tau\Big\} \ \ (\tau > 0).$$

This led to the famous "law of the iterated logarithm," which is the result of researches related to the refining of the strong Borel law of large numbers [117, Sec. 1]. Theorems 1.3 and 1.4 and Corollary 1.5 are the source of many researches and generalizations. They were applied in various fields of analysis (see, e.g., [117], which contains different variants of the Khintchine inequality in the case of martingale sequences, results related to the search of the best constants in these inequalities, and many other interesting information). Such inequalities are very important in the present book. Inequalities (1.12) is the origin of various researches. For example, [69] is devoted to the following problem: For which convex increasing functions Φ on $[0,\infty)$, $\Phi(0) = 0$, does the inequality

$$\int_0^1 \Phi\Big(\sum_{k=1}^{n} \sigma_i r_i(t)\Big)\, dt \leq \mathbb{E}\Phi\Big(\sum_{k=1}^{n} \eta_i\Big)$$

hold for all independent symmetrically distributed random values η_i $(i = 1, 2, \dots, n)$ such that $\mathbb{E}\eta_i^2 = \sigma_i^2$?

The result obtained in Remark 1.4 was proved in [134]. A vector-valued version of the Khintchine inequality proved in Corollary 1.6 was applied for studying spaces of measurable functions (see, e.g., [97]). Proposition 1.2 states that Rademacher functions form a Sidon–Zygmund system in the narrow sense. This result was proved in [80]. Theorem 1.5 was proved by Borrel (see [46]) in 1979 (the monograph was not published). Its proof is based on Lemma 1.2, which was obtained in [37] (in the description of these results we follow [97, Theorem 1.e.13 and Lemma 1.e.14]). The important Proposition 1.5 was proved in the monograph [124].

Chapter 2

RADEMACHER SUMS IN L_p-SPACES AND THEIR DISTRIBUTION

Estimates of L_p-norms of the previous chapter (first of all, of inequalities (1.15) and (1.20)) include constants depending on p; that is why on their basis we can claim only that

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_p \le C(a)\sqrt{p} \ \ (p \ge 1) \quad \text{if} \quad (a_k) \in \ell_2. \tag{2.1}$$

Here more exact results will be obtained, with which help we can define, for every sequence of coefficients $(a_k)_{k=1}^{\infty}$, the order of the norm $\Big\| \sum\limits_{k=1}^{\infty} a_k r_k \Big\|_p$ as $p \to \infty$. The Peetre $\mathcal{K}$-functional $\kappa_a(t) := \mathcal{K}(t, a; \ell_1, \ell_2)$ constructed on the Banach couple (ℓ_1, ℓ_2) will play the central role. Recall its definition: if (X_0, X_1) is a Banach couple $x \in X_0 + X_1$ and $t > 0$, then

$$\mathcal{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : \, x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

First of all, we must "decode" the functional $\kappa_a(t)$, finding quite simple equivalent expressions for it. Let us begin from its functional analogue (we need it in the future).

Proposition 2.1. *For arbitrary $f \in L_1(0,\infty) + L_2(0,\infty)$ and any $t > 0$, we have*

$$\begin{aligned} \frac{1}{4}\Bigg\{ \int_0^{t^2} f^*(s)ds + t\bigg(\int_{t^2}^{\infty} f^*(s)^2 ds \bigg)^{1/2} \Bigg\} &\le \mathcal{K}(t, f; L_1, L_2) \\ &\le \int_0^{t^2} f^*(s)ds + t\bigg(\int_{t^2}^{\infty} f^*(s)^2 ds \bigg)^{1/2}. \end{aligned} \tag{2.2}$$

Proof. For any $t > 0$, denote $E = \{s > 0 : |f(s)| \ge f^*(t^2)\}$, $f_0(s) = f(s)\chi_E(s)$, and $f_1(s) = f(s) - f_0(s)$. Without loss of generality we can assume that $m\{s > 0 : \, f(s) = \tau\} = 0$ for any $\tau \in \mathbb{R}$. Then the function $|f_0|$ is equimeasurable with the function $f^*\chi_{(0,t^2)}$, and $|f_1|$ is equimeasurable with $f^*\chi_{(t^2,\infty)}$. Therefore, we have

$$\|f_0\|_1 + t\|f_1\|_2 = \int_0^{t^2} f^*(s)\, ds + t\bigg(\int_{t^2}^{\infty} f^*(s)^2\, ds \bigg)^{1/2};$$

this immediately implies the right-hand inequality in (2.2).

To prove the inverse inequality, assume that $f = g + h$, where $g \in L_1$ and $h \in L_2$. For any $\varepsilon > 0$, by the definition of rearrangement, we have

$$m\{s > 0 : \, |g(s)| > g^*(u/2 - \varepsilon)\} \le u/2 - \varepsilon$$

and

$$m\{s > 0 : \, |h(s)| > h^*(u/2 - \varepsilon)\} \le u/2 - \varepsilon.$$

Therefore, the following inequality holds:

$$m\{s > 0 : \, |f(s)| > g^*(u/2 - \varepsilon) + h^*(u/2 - \varepsilon)\} \le u - 2\varepsilon < u,$$

where

$$f^*(u) \le g^*(u/2 - \varepsilon) + h^*(u/2 - \varepsilon).$$

Since the rearrangement is continuous from the left and $\varepsilon > 0$ is arbitrary, it follows that

$$f^*(u) \le g^*(u/2) + h^*(u/2), \quad u > 0. \tag{2.3}$$

Therefore, applying the Cauchy–Bunyakovskii inequality, we obtain the inequality

$$\begin{aligned} \|g\|_1 + t\|h\|_2 &\ge \frac{1}{2}\left(\int_0^{t^2} g^*(u/2)\,du + t\left(\int_0^{t^2} \big(h^*(u/2)\big)^2\,du\right)^{1/2}\right) \\ &\ge \frac{1}{2}\left(\int_0^{t^2} g^*(u/2)\,du + \int_0^{t^2} h^*(u/2)\,du\right) \ge \frac{1}{2}\int_0^{t^2} f^*(u)\,du. \end{aligned} \tag{2.4}$$

Moreover,

$$\begin{aligned} t^2\int_{t^2}^{\infty} \big(g^*(u/2)\big)^2\,du &\le \int_0^{t^2} g^*(u/2)\,du \int_{t^2}^{\infty} g^*(u/2)\,du \\ &\le \left(\int_0^{\infty} g^*(u/2)\,du\right)^2 = 4\|g\|_1^2, \end{aligned}$$

and, due to (2.3),

$$\|g\|_1 + t\|h\|_2 \ge \frac{t}{2}\left(\int_{t^2}^{\infty} \big(g^*(u/2)\big)^2\,du\right)^{1/2} + \frac{t}{2}\left(\int_{t^2}^{\infty} \big(h^*(u/2)\big)^2\,du\right)^{1/2} \ge \frac{t}{2}\left(\int_{t^2}^{\infty} f^*(u)^2\,du\right)^{1/2}.$$

From here and from (2.4), by the definition of $\mathcal{K}$-functional, we obtain the left-hand inequality in (2.2). $\square$

In order to obtain the same result for $\kappa_a(t)$, we need the following auxiliary assertion, where

$$f_a(t) = \sum_{k=1}^{\infty} a_k\, \chi_{(k-1,k]}(t) \ \ (t > 0) \tag{2.5}$$

for any sequence $a = (a_k)_{k=1}^{\infty} \in \ell_2$.

Lemma 2.1. *For any $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $t > 0$, we have*

$$\mathcal{K}(t, f_a; L_1, L_2) = \kappa_a(t). \tag{2.6}$$

Proof. First of all, one can check that

$$\|f_a\|_1 = \|a\|_{\ell_1} \quad \text{and} \quad \|f_a\|_2 = \|a\|_{\ell_2}. \tag{2.7}$$

If $a = b + c$, where $c \in \ell_2$, then

$$\|b\|_{\ell_1} + t\|c\|_{\ell_2} = \|f_b\|_1 + t\|f_c\|_2$$

for the functions $f_b = \sum_{k=1}^{\infty} b_k \chi_{(k-1,k]}$ and $f_c = \sum_{k=1}^{\infty} c_k \chi_{(k-1,k]}$. Since $f_b + f_c = f_a$, it follows that

$$\mathcal{K}(t, f_a; L_1, L_2) \le \mathcal{K}(t, a; \ell_1, \ell_2).$$

On the contrary, let $f_a = g + h$, where $g \in L_1$, $h \in L_2$. One can check that the mean operator

$$Qx(t) = \sum_{k=1}^{\infty} a_k(x)\,\chi_{(k-1,k]}(t) \ \ (t > 0),$$

where $a_k(x) = \int\limits_{k-1}^{k} x(s)\,ds$, is a projector in L_1 and in L_2 with norm 1. Therefore, due to (2.7), we have

$$\|g\|_1 + t\|h\|_2 \geq \|Qg\|_1 + t\|Qh\|_2 = \|a(g)\|_{\ell_1} + t\|a(h)\|_{\ell_2}$$

where $a(x) = (a_k(x))_{k=1}^{\infty}$. Moreover, since $f_a = Qf_a = Qg + Qh$, it follows that $a = a(g) + a(h)$. The latter inequality implies that

$$\mathcal{K}(t, a; \ell_1, \ell_2) \leq \mathcal{K}(t, f_a; L_1, L_2);$$

therefore, (2.6) is proved. □

Proposition 2.2. *For any $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $t > 0$, the following inequality holds:*

$$\frac{1}{4}\left\{\sum_{i=1}^{[t^2]} a_i^* + t\left[\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\right]^{1/2}\right\} \leq \kappa_a(t) \leq \left\{\sum_{i=1}^{[t^2]} a_i^* + t\left[\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\right]^{1/2}\right\},$$

where $(a_i^*)_{i=1}^{\infty}$ is a non-increasing permutation of the sequence $(|a_n|)_{n=1}^{\infty}$.

Proof. If $0 < t < 1$, then due to inequality $\|a\|_2 \leq \|a\|_1$, $\kappa_a(t) = t\|a\|_2$. Then (2.2) holds.

Let $a = (a_k)_{k=1}^{\infty}$ and $t \geq 1$. Since the spaces ℓ_1 and ℓ_2 are symmetric, it follows from the definition of the $\mathcal{K}$-functional that $\kappa_{|a_\pi|}(t) = \kappa_a(t)$, where π is an arbitrary permutation of the positive integers and $a_\pi = (a_{\pi(k)})_{k=1}^{\infty}$. Therefore, we can assume that $a_k = a_k^*$.

For a fixed t, introduce $b^t = (b_k^t)$, where $b_k^t = a_k$ if $k \leq [t^2]$ and $b_k^t = 0$ if $k > [t^2]$. Moreover, let $c^t = a - b^t$. Then, by the definition of the $\mathcal{K}$-functional, we obtain the right-hand inequality in (2.2). The left-hand inequality follows from Proposition 2.1 and Lemma 2.1. Indeed, if f_a is defined by (2.5), then

$$\kappa_a(t) \geq \kappa_a(\sqrt{[t^2]}) = \mathcal{K}(\sqrt{[t^2]}, f_a; L_1, L_2) \geq \frac{1}{4}\left\{\int\limits_0^{[t^2]} f_a^*(s)\,ds + t\Big(\int\limits_{[t^2]}^{\infty} f_a^*(s)^2\,ds\Big)^{1/2}\right\}$$

$$= \frac{1}{4}\left\{\sum_{i=1}^{[t^2]} a_i^* + t\Big(\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\Big)^{1/2}\right\}.$$

□

In Chap. 8, we need the following properties of the functional $\kappa_a(t)$, which follow from relation (2.2).

Corollary 2.1. *The function $\kappa_a(t)$ is continuous and increases as $t \geq 0$ and*

(1) $\lim\limits_{t\to 0+} \kappa_a(t) = 0$;

(2) $\frac{1}{4}\|a\|_1 \leq \lim\limits_{t\to +\infty} \kappa_a(t) \leq \|a\|_1$

for any sequence $a = (a_k)_{k=1}^{\infty} \in \ell_2$.

Further, we repeatedly need one more approximation of the functional $\kappa_a(t)$. For arbitrary $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $t \in \mathbb{N}$, we define the norm

$$\|a\|_{Q(t)} = \sup\left\{\sum_{j=1}^{t}\left(\sum_{n\in A_j} a_n^2\right)^{1/2}\right\}, \tag{2.8}$$

where the supremum is taken over all partitions $\{A_j\}_{j=1}^t$ of the positive integers $\mathbb{N}$.

Lemma 2.2. *If $a = (a_k)_{k=1}^\infty \in \ell_2$ and $t^2 \in \mathbb{N}$, then*

$$\|a\|_{Q(t^2)} \le \kappa_a(t) \le \sqrt{2}\,\|a\|_{Q(t^2)}. \tag{2.9}$$

Proof. First of all, it follows from the definition of $\|\cdot\|_{Q(t)}$ that $\|a\|_{Q(t^2)} \le \|a\|_1$ and $\|a\|_{Q(t^2)} \le t\|a\|_2$. Indeed, the former inequality is obvious and to prove the latter one, we note that due to the Cauchy–Bunyakovskii inequality, we have

$$\sum_{j=1}^{t^2}\left(\sum_{n\in A_j} a_n^2\right)^{1/2} \le t\left(\sum_{j=1}^{t^2}\sum_{n\in A_j} a_n^2\right)^{1/2} = t\|a\|_2,$$

where $\{A_j\}_{j=1}^{t^2}$ is an arbitrary partition of the positive integers. That is why

$$\begin{aligned}\kappa_a(t) &= \inf\{\|b\|_1 + t\|c\|_2 :\ b+c=a, b\in\ell_1, c\in\ell_2\}\\ &\ge\ \inf\{\|b\|_{Q(t^2)} + \|c\|_{Q(t^2)} :\ b+c=a, b\in\ell_1, c\in\ell_2\} \ge \|a\|_{Q(t^2)};\end{aligned}$$

thus, the left-hand side of (2.9) is proved.

To prove the right-hand side of (2.9) we need the following relation:

$$\kappa_a(t) = \sup\left\{\sum_{k=1}^\infty a_k b_k :\ b = (b_k)_{k=1}^\infty \in \ell_2\,,\ \mathcal{J}_{\infty,2}(t^{-1}, b) \le 1\right\}, \tag{2.10}$$

where $\mathcal{J}_{\infty,2}(s,b) := \max\{\|b\|_\infty, s\|b\|_2\}$ $(s>0)$. To prove (2.10), we denote by ℓ_2^t the space ℓ_2 with the norm $\kappa_a(t)$. Then, by the Hahn–Banach theorem, we have

$$\kappa_a(t) = \sup\{|f(a)| :\ f\in(\ell_2^t)^*\,,\ \|f\|_{(\ell_2^t)^*} \le 1\}.$$

Therefore, it suffices to prove that the dual space $(\ell_2^t)^*$ can be described as the set of all functionals of the form

$$f(a) = \sum_{k=1}^\infty a_k b_k \quad (a\in\ell_2) \tag{2.11}$$

and $\mathcal{J}_{\infty,2}(1/t, b) = \|f\|_{(\ell_2^t)^*}$.

Indeed, let $f \in (\ell_2^t)^*$. If $a\in\ell_1$, then $|f(a)| \le \|f\|_{(\ell_2^t)^*}\|a\|_1$. Hence, f is presented as (2.11) and we obtain $\|b\|_\infty \le \|f\|_{(\ell_2^t)^*}$ for $b = (b_k)_{k=1}^\infty$. Similarly, we have $\|b\|_2 \le t\|f\|_{(\ell_2^t)^*}$. Therefore, $\mathcal{J}_{\infty,2}(1/t,b) \le \|f\|_{(\ell_2^t)^*}$.

On the contrary, let $a\in\ell_2$. Then, if $a = c+d$, $c\in\ell_1$, and $d\in\ell_2$, then

$$\begin{aligned}\Big|\sum_{k=1}^\infty a_k b_k\Big| &\le \Big|\sum_{k=1}^\infty c_k b_k\Big| + \Big|\sum_{k=1}^\infty d_k b_k\Big| \le \|b\|_\infty \sum_{k=1}^\infty |c_k|\\ &+ \frac{1}{t}\Big(\sum_{k=1}^\infty b_k^2\Big)^{1/2}\cdot t\Big(\sum_{k=1}^\infty d_k^2\Big)^{1/2} \le \mathcal{J}_{\infty,2}(1/t,b)(\|c\|_1 + t\|d\|_2).\end{aligned}$$

Passing to the greatest lower bound over all admissible representations of a, we obtain that the following expression holds for the functional f defined by relation (2.11):

$$\|f\|_{(\ell_2^t)^*} \le \mathcal{J}_{\infty,2}(1/t,b).$$

Hence, (2.10) is proved.

Due to (2.10), for any $\delta>0$ there exists a sequence $b\in\ell_2$ such that

$$(1-\delta)\,\kappa_a(t) \le \sum_{k=1}^\infty a_k b_k \quad\text{and}\quad \mathcal{J}_{\infty,2}(t^{-1}, b) \le 1.$$

Choose $n_0, n_1, n_2, \ldots, n_{t^2} \in \{0, 1, \ldots, \infty\}$ by induction as follows: if $0 = n_0 < n_1 < \ldots < n_m$ are already found, then

$$n_{m+1} = 1 + \sup\Big\{k : \sum_{n=n_m+1}^{k} b_n^2 \le 1\Big\}.$$

Since $\|b\|_\infty \le 1$, we have $\sum\limits_{n=n_m+1}^{n_{m+1}} b_n^2 \le 2$. From $\|b\|_2 \le t$, it follows that $n_{t^2} = \infty$. Therefore,

$$(1-\delta)\,\kappa_a(t) \le \sum_{k=1}^{\infty} a_k b_k \le \sum_{m=1}^{t^2} \Big(\sum_{n=n_{m-1}+1}^{n_m} b_n^2\Big)^{1/2} \Big(\sum_{n=n_{m-1}+1}^{n_m} a_n^2\Big)^{1/2} \le \sqrt{2}\,\|a\|_{Q(t^2)}.$$

Since the last holds for all $\delta > 0$, this lemma is proved. □

Now we can formulate and comment on one of the most important results of this chapter.

Theorem 2.1. *There exists a universal constant $\gamma > 0$ such that for all $t \ge 1$ and all $a = (a_k)_{k=1}^\infty \in \ell_2$ the following inequality holds:*

$$\gamma \kappa_a(\sqrt{t}) \le \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_t \le \kappa_a(\sqrt{t}). \tag{2.12}$$

Before we prove this statement, we make sure that inequalities (2.2) and (2.12) are able to give an exact asymptotics of the norm $\Big\|\sum\limits_{k=1}^{\infty} a_k r_k\Big\|_t$ for different particular sequences of coefficients $(a_k)_{k=1}^\infty$. Consider three examples: $(a)\, a_k = 1/k$ $(k = 1, 2, \ldots)$, $(b)\, a_k = 2^{-k}$ $(k = 1, 2, \ldots)$, and $(c)\, a_k = 1/\sqrt{n}$ $(k = 1, 2, \ldots, n)$ and $a_k = 0$ $(k > n)$. The Khintchine inequality (see (1.10)) implies only estimate (2.1) for all three cases. Using inequalities (2.2) and (2.12), we obtain more complete information about the asymptotic behavior of this quantity. This information is different for these sequences. To be exact, simple calculations show that

$$\text{(a)} \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_t \asymp \ln(1+t), \quad \text{(b)} \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_t \asymp 1,$$
$$\text{(c)} \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_t \asymp \sqrt{\min(n,t)}.$$

To prove Theorem 2.1, we need the Paley–Zygmund inequality, which plays an important role in the systems of independent functions studied (see, e.g., [82]).

Lemma 2.3. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, $f \in L_2(\Omega, \Sigma, \mathbb{P})$, and $f \ge 0$. Then for any $0 < \lambda < 1$ we have*

$$\mathbb{P}\{f \ge \lambda \mathbb{E}(f)\} \ge (1-\lambda)^2 \frac{(\mathbb{E}(f))^2}{\mathbb{E}(f^2)}. \tag{2.13}$$

In particular, if m is the Lebesgue measure on $[0,1]$ and $(a_k)_{k=1}^\infty \in \ell_2$, then

$$m\left\{t \in [0,1] : \Big|\sum_{k=1}^{\infty} a_k r_k(t)\Big|^2 \ge \lambda \sum_{k=1}^{\infty} a_k^2\right\} \ge \frac{1}{9}(1-\lambda)^2. \tag{2.14}$$

Proof. Define the random variable g as follows:

$$g(\omega) = f(\omega) \ \text{ if } \ f(\omega) \ge \lambda \mathbb{E}(f) \quad \text{and} \quad g(\omega) = 0 \ \text{ if } \ f(\omega) < \lambda \mathbb{E}(f).$$

Due to the integral Cauchy–Bunyakovskii inequality, we have

$$(\mathbb{E}(g))^2 \le \mathbb{E}(g^2)\mathbb{P}\{g \ne 0\} \le \mathbb{E}(f^2)\mathbb{P}\{f \ge \lambda \mathbb{E}(f)\}.$$

Moreover, $\mathbb{E}(f) \le \mathbb{E}(g) + \lambda\mathbb{E}(f)$. Therefore,

$$(1-\lambda)^2(\mathbb{E}(f))^2 \le \mathbb{E}(f^2)\mathbb{P}\{f \ge \lambda\mathbb{E}(f)\}$$

and inequality (2.13) is proved.

In order to obtain inequality (2.14), we consider the function

$$f(t) := \Big|\sum_{i=1}^{n} a_i r_i(t)\Big|^2.$$

Then $\mathbb{E}(f) = \sum_{k=1}^{n} a_k^2$ and, due to (1.10) for $p = 4$, we have $(\mathbb{E}(f))^2 \ge \frac{1}{9}\mathbb{E}(f^2)$. Therefore, (2.14) is a particular case of (2.13). □

Proof of Theorem 2.1. The right-hand side of the inequality is a corollary of the definition of the $\mathcal{K}$-functional and the Khintchine inequality. Indeed, for arbitrary $\varepsilon > 0$, we find $b \in \ell_1$ and $c \in \ell_2$ such that

$$\|b\|_1 + \sqrt{t}\|c\|_2 \le \kappa_a(\sqrt{t}) + \varepsilon.$$

Then, applying inequality (1.15) (taking into account the value of the constant), we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_t &\le \Big\|\sum_{k=1}^{\infty} b_k r_k\Big\|_t + \Big\|\sum_{k=1}^{\infty} c_k r_k\Big\|_t \\ &\le \sum_{k=1}^{\infty}|b_k| + \sqrt{t}\left(\sum_{k=1}^{\infty} c_k^2\right)^{1/2} \le \kappa_a(\sqrt{t}) + \varepsilon.\end{aligned}$$

First we prove the left-hand side for $t \in \mathbb{N}$. By the definition of $Q(t)$, we choose pairwise disjoint sets $A_1, \dots, A_t$ of the positive integers such that

$$\|a\|_{Q(t)} \le \frac{3}{2}\left\{\sum_{j=1}^{t}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right\}. \tag{2.15}$$

Using Chebyshev's inequality and the independence of Rademacher functions (see Proposition 1.1), we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_t^t &= \Big\|\sum_{j=1}^{t}\sum_{k\in A_j} a_k r_k\Big\|_t^t \\ &\ge 2^{-t}\left\{\sum_{j=1}^{t}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right\}^t m\left(\bigcap_{j=1}^{t}\left\{\sum_{k\in A_j} a_k r_k \ge \frac{1}{2}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right\}\right) \\ &= 2^{-t}\left\{\sum_{j=1}^{t}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right\}^t \prod_{j=1}^{t} m\left(\sum_{k\in A_j} a_k r_k \ge \frac{1}{2}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right).\end{aligned}$$

Since, due to the symmetry of the Rademacher functions distribution and the Paley–Zygmund inequality (see (2.14)), we have

$$m\left(\sum_{k\in A_j} a_k r_k \ge \frac{1}{2}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right) = \frac{1}{2}m\left(\Big|\sum_{k\in A_j} a_k r_k\Big|^2 \ge \frac{1}{4}\sum_{k\in A_j} a_k^2\right) \ge \frac{1}{32},$$

it follows from (2.15) and Lemma 2.2 that

$$\left\|\sum_{k=1}^{\infty} a_k r_k\right\|_t \geq \frac{1}{64}\sum_{j=1}^{t}\left(\sum_{k\in A_j} a_k^2\right)^{1/2} \geq \frac{1}{96\sqrt{2}}\kappa_a(\sqrt{t}).$$

If $t \geq 1$ is arbitrary, then, taking into account the concavity of the $\mathcal{K}$-functional and the simple inequality $\sqrt{t} \leq 2\sqrt{[t]}$, we obtain

$$\frac{1}{192\sqrt{2}}\kappa_a(\sqrt{t}) \leq \frac{1}{96\sqrt{2}}\kappa_a(\sqrt{[t]}) \leq \left\|\sum_{k=1}^{\infty} a_k r_k\right\|_{[t]} \leq \left\|\sum_{k=1}^{\infty} a_k r_k\right\|_t$$

and Theorem 2.1 is proved. □

The next important result allows us to find a distribution of functions represented as the Rademacher series. It will play an important role in the study of subspaces of symmetric spaces generated by this system.

Theorem 2.2. *There exists a universal constant $C > 0$ such that for all $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $t \geq 0$ the following inequalities hold:*

$$m\left\{\sum_{k=1}^{\infty} a_k r_k \,>\, \kappa_a(t)\right\} \leq e^{-t^2/2} \tag{2.16}$$

$$m\left\{\sum_{k=1}^{\infty} a_k r_k \,>\, C^{-1}\kappa_a(t)\right\} \geq C^{-1}e^{-Ct^2}. \tag{2.17}$$

In Corollary 1.5, using the Khintchine inequality, we obtained an exponential estimate for polynomial distributions with respect to the Rademacher system (see inequality (1.13)). We need a more exact version of this corollary in the proof of Theorem 2.2.

Lemma 2.4. *If $S_n = \sum\limits_{k=1}^{n} a_k r_k$, then for arbitrary $u > 0$, we have*

$$m\{t \in [0,1] :\, S_n(t) > u\} \leq \exp\left(-\frac{u^2}{2\sum\limits_{k=1}^{n} a_k^2}\right). \tag{2.18}$$

Proof. Due to Chebyshev's inequality and (1.2), from Proposition 1.1, for any $\lambda > 0$, we have

$$\begin{aligned} m\{t \in [0,1] :\, S_n(t) > u\} &= m\{t \in [0,1] :\, \exp(\lambda S_n(t)) > \exp(\lambda u)\} \leq e^{-\lambda u}\mathbb{E}(e^{\lambda S_n}) \\ &= e^{-\lambda u}\prod_{k=1}^{n}\mathbb{E}(e^{\lambda a_k r_k}) = e^{-\lambda u}\prod_{k=1}^{n}\cosh(\lambda a_k). \end{aligned}$$

Since $\cosh x := 2^{-1}(e^x + e^{-x}) \leq e^{x^2/2}$ (it suffices to expand each of these functions into a Taylor series and compare coefficients with respect to equal degrees), we have

$$m\{t \in [0,1] :\, S_n(t) > u\} \leq e^{-\lambda u}\prod_{k=1}^{n}\exp\left(\frac{\lambda^2 a_k^2}{2}\right) = \exp\left(\frac{\lambda^2\sum\limits_{k=1}^{n} a_k^2}{2} - \lambda u\right).$$

One can easily check that the minimum of the right-hand side is achieved for $\lambda_0 = u/\sum\limits_{k=1}^{n} a_k^2$. Substituting this value, we obtain inequality (2.18). □

Proof of Theorem 2.2. First we prove inequality (2.16). By the definition of the $\mathcal{K}$-functional, for arbitrary $\delta > 0$, we find a decomposition $a = b + c$, where $b, c \in \ell_2$, such that

$$(1+\delta)\kappa_a(t) \ > \ \|b\|_1 + t\|c\|_2.$$

Then

$$m\left\{\sum_{k=1}^{\infty} a_k r_k \ > \ (1+\delta)\kappa_a(t)\right\} \le m\left\{\sum_{k=1}^{\infty} b_k r_k \ > \ \|b\|_1\right\} + m\left\{\sum_{k=1}^{\infty} c_k r_k \ > \ t\|c\|_2\right\} \le e^{-t^2/2}.$$

Here we used Lemma 2.4 and the fact that (see Remark 1.5)

$$m\left\{\sum_{k=1}^{\infty} b_k r_k \ > \ \|b\|_1\right\} = 0.$$

Passing to the limit as $\delta \to 0$, we obtain (2.16).

To prove inequality (2.17), assume that $t \in \mathbb{N}$. For fixed $\delta > 0$, we find pairwise disjoint $A_1, A_2, \dots, A_{t^2}$ such that

$$\bigcup_{j=1}^{t^2} A_j = \mathbb{N} \quad \text{and} \quad \|a\|_{Q(t^2)} \le (1+\delta)\sum_{j=1}^{t^2}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}.$$

Then, due to Lemma 2.2 and the independence of Rademacher functions (Proposition 1.1), we have

$$\begin{aligned}
m\left\{\sum_{k=1}^{\infty} a_k r_k \ > \ \frac{1}{2}\kappa_a(t)\right\} &\ge m\left\{\sum_{k=1}^{\infty} a_k r_k \ > \ \frac{1}{\sqrt{2}}\|a\|_{Q(t^2)}\right\} \\
&\ge m\left\{\sum_{j=1}^{t^2}\sum_{k\in A_j} a_k r_k > \frac{1}{\sqrt{2}}(1+\delta)\sum_{j=1}^{t^2}\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right\} \\
&\ge \prod_{j=1}^{t^2} m\left\{\sum_{k\in A_j} a_k r_k > \frac{1}{\sqrt{2}}(1+\delta)\left(\sum_{k\in A_j} a_k^2\right)^{1/2}\right\}.
\end{aligned}$$

Using (2.14) and taking into account the symmetry of the distribution of the functions $\sum\limits_{k\in A_j} a_k r_k$ $(j = 1, 2, \dots, t^2)$, we obtain that

$$m\left\{\sum_{k=1}^{\infty} a_k r_k \ > \ \frac{1}{2}\kappa_a(t)\right\} \ge \left(\frac{1}{18}\Big(1 - \frac{1}{2}(1+\delta)^2\Big)^2\right)^{t^2}.$$

Passing to the limit as $\delta \to 0$, we get

$$m\left\{\sum_{k=1}^{\infty} a_k r_k \ > \ \frac{1}{2}\kappa_a(t)\right\} \ge \exp\big(-(\ln 72)t^2\big).$$

Therefore, (2.17) is proved if $t \in \mathbb{N}$.

In the case where $t \ge 1$, writing (2.17) for the integer $[t] + 1$ and applying the inequalities

$$\kappa_a(t) \le \kappa_a([t]+1) \quad \text{and} \quad ([t]+1)^2 \le 4t^2,$$

we obtain that (2.17) holds with $C = 4\ln 72$. If $t < 1$, then $\kappa_a(t) = t\|a\|_2$ and, due to the Paley–Zygmund inequality, we have

$$m\left\{\sum_{k=1}^{\infty} a_k r_k > \frac{1}{2}\kappa_a(t)\right\} = m\left\{\sum_{k=1}^{\infty} a_k r_k > \frac{t}{2}\|a\|_2\right\} \ge \frac{1}{18}\left(1 - \frac{t^2}{4}\right)^2 \ge \frac{1}{32}.$$

Thus, (2.17) holds for all $t > 0$ with the constant $C = 32$. $\square$

The proved theorem allows us to find exact estimates from above and from below for non-increasing rearrangements of arbitrary Rademacher sums.

Corollary 2.2. *There exist universal constants $\beta \in (0,1)$, $C_1 > 0$, and $C_2 > 0$ such that for all $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $0 < u \le 1$, we have*

$$x_a^*(u) \le C_1 \kappa_a(\ln^{1/2}(e/u)) \tag{2.19}$$

and

$$x_a^*(\beta u) \ge C_2^{-1} \kappa_a(\ln^{1/2}(e/u)), \tag{2.20}$$

where $x_a(s) = \sum\limits_{k=1}^{\infty} a_k r_k(s)$.

Proof. Due to the symmetry of the distribution of Rademacher sums and relation (2.16), we have

$$m\{s \in [0,1] : |x_a(s)| > \kappa_a(t)\} = 2m\{s \in [0,1] : x_a(s) > \kappa_a(t)\} < e \cdot e^{-t^2/2}.$$

By the definition of non-increasing rearrangements of measurable functions, substituting $u = e \cdot e^{-t^2/2}$, we obtain the inequality

$$x_a^*(u) \le \kappa_a(\sqrt{2}\ln^{1/2}(e/u)) \le \sqrt{2}\kappa_a(\ln^{1/2}(e/u)),$$

i.e., inequality (2.19) is proved with $C_1 = \sqrt{2}$.

Similar reasoning and inequality (2.17) lead to the relation

$$x_a^*\big(C^{-1}e^{-Ct^2}\big) \ge C^{-1}\kappa_a(t) \quad (t \ge 0)$$

where $C > 1$ is universal constant. Due to the concavity of the $\mathcal{K}$-functional, this yields

$$x_a^*(v) \ge C^{-1}\kappa_a(C^{-1/2}\ln^{1/2}(1/(Cv))) \ge C^{-3/2}\kappa_a(\ln^{1/2}(1/(Cv))) \quad (0 < v \le C^{-1}),$$

i.e.,

$$x_a^*(u/(eC)) \ge C^{-3/2}\kappa_a(\ln^{1/2}(e/u)), \quad 0 < u \le 1.$$

Thus, we obtain inequality (2.20) with $\beta = (Ce)^{-1}$ and $C_2 = C^{3/2}$. $\square$

Comments and References. The results of this chapter are mostly related to the behavior of Rademacher sums in L_p-spaces and essentially refine classical Khintchine inequalities. Their advantage is that they allow us to determine the asymptotic behavior of L_p-norms of Rademacher polynomials as $p \to \infty$ (see examples before Lemma 2.3).

Beginning from this chapter, we widely use the theory of interpolation of operators and, in particular the Peetre $\mathcal{K}$-functional introduced in 1963 (see [114]). The $\mathcal{K}$-functional corresponding to the Banach couple (ℓ_1, ℓ_2) of sequence spaces plays an important role in questions related to the Rademacher system. Propositions 2.1 and 2.2 are the particular cases of the theorem proved by Holmstedt in [76]. Theorem 2.1 due to Hitchenko (see [75]) allows us to find the order of the norm $\Big\| \sum\limits_{k=1}^{\infty} a_k r_k \Big\|_p$ as $p \to \infty$ for any fixed sequence of coefficients $(a_k)_{k=1}^{\infty} \in \ell_2$. Lemmas 2.3 and 2.2 are used to prove this theorem. The former contains the Paley–Zygmund inequality, which is fundamental for the study of systems of independent functions (see, e.g., [82]). The latter finds a useful equivalent expression for the $\mathcal{K}$-functional corresponding to the Banach couple (ℓ_1, ℓ_2). Using Theorem 2.2, we find (up to equivalence) a distribution of Rademacher sums. This important fact was obtained by Montgomery-Smith (see [109]). It and Corollary 2.2 proved on the basis of this fact play further an important role.

Chapter 3

RADEMACHER SYSTEM IN SYMMETRIC SPACES "FAR" FROM L_∞

We proceed to the study of the subspace $\mathcal{R}(X)$ of symmetric space X consisting of all measurable functions $g\colon [0,1] \to \mathbb{R}$, $g \in X$, represented as $g(t) = \sum\limits_{k=1}^{\infty} a_k r_k(t)$ (the series converges a.e.), where $r_k(t)$ are Rademacher functions. It can be naturally described as the space of sequences. Let E_X consist of all sequences $a = (a_k)_{k=1}^{\infty}$ such that the function $x_a := \sum\limits_{k=1}^{\infty} a_k r_k$ belongs to X. It is easy to check that the functional

$$\|a\|_{E_X} := \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X \tag{3.1}$$

is a norm. In particular, if $\|a\|_{E_X} = 0$, then the relation $a = 0$ is a corollary of the orthogonality of Rademacher functions (see also inequality (1.25) from the proof of Corollary 1.7). Show that E_X is complete.

Let a sequence $\{a^n\}_{n=1}^{\infty}$, $a^n = (a_1^n, a_2^n, \ldots)$, be a Cauchy sequence in E_X. This means that the sequence of functions $\{x_a^n\}$, $x_a^n = \sum\limits_{k=1}^{\infty} a_k^n r_k$, is a Cauchy sequence in X. Therefore, there exists a function x from X such that $x_a^n \to x$ as $n \to \infty$ in X and in L_1 (see [92, Theorem 2.4.1]).

On the other hand, due to the Khintchine L_1-inequality from Theorem 1.4 the sequence $\{a^n\} = \{a_k^n\}_{k=1}^{\infty}$ is a Cauchy sequence in ℓ_2, hence, it has a limit there. Denote it by $a = \{a_k\}_{k=1}^{\infty}$. If $x_a = \sum\limits_{k=1}^{\infty} a_k r_k$, then

$$\|x_a^n - x_a\|_1 \le \|x_a^n - x_a\|_2 = \|a^n - a\|_{\ell_2}.$$

Therefore, $x_a^n \to x_a$ in L_1. As a result, $x = x_a$ from which $x_a^n \to x_a$ in X or equivalent $a^n \to a$ in E_X. Therefore, the completeness of the space E_X (and, therefore, the closure of $\mathcal{R}(X)$ in symmetric space X) is proved.

Note that, due to Corollary 1.7, if the space E_X contains a sequence $a = (a_k)_{k=1}^{\infty}$, then it contains any sequence $a_\pi = (a_{\pi(k)})$, where π is an arbitrary permutation of the set of positive integers. Finally, we come to the conclusion that every symmetric space X corresponds to the symmetric space of sequences E_X, which is isometric to $\mathcal{R}(X)$.

As we will see, properties of $\mathcal{R}(X)$ essentially depend on the order of "closeness" of X to the space L_∞. The space G is the separable part of the Orlicz space L_{N_2} constructed with respect to the function $N_2(t) = e^{t^2} - 1$. It will be the natural "boundary".

Recall that, by Theorem 1.4, the function

$$f(t) = \sum_{k=1}^{\infty} a_k r_k(t) \tag{3.2}$$

is well defined and belongs to the space $L_p[0,1]$ for every $1 \le p < \infty$ if the sequence of coefficients is square summable, i.e.,

$$\|a\|_2^2 = \sum_{k=1}^{\infty} a_k^2 \ < \ \infty. \tag{3.3}$$

We show that all functions of such form are summable in a stronger exponential sense, i.e., $\exp(f(t)^2) \in L_1[0,1]$. We prefer to formulate and prove the result in terms of Orlicz function spaces.

Theorem 3.1. *Function* (3.2) *belongs to the space G if Condition* (3.3) *holds.*

Proof. Using the Taylor expansion

$$\exp(u^2 z^2) = \sum_{k=0}^{\infty} \frac{u^{2k}}{k!} z^{2k},$$

inequality (1.11), and the estimate $B_{2m} \leq m^m$, we obtain the inequality

$$\int_0^1 (\exp(u^2 f(t)^2) - 1)\, dt = \sum_{k=1}^{\infty} \frac{u^{2k}}{k!} \int_0^1 f(t)^{2k}\, dt \leq \sum_{k=1}^{\infty} \frac{u^{2k}}{k!} k^k \|a\|_2^{2k}.$$

If $u \leq (\sqrt{2e}\|a\|_2)^{-1}$, then, due to (1.14), the sum on the right-hand side of the last inequality does not exceed $(2\pi)^{-1/2} \sum_{k=1}^{\infty} 2^{-k} = (2\pi)^{-1/2} < 1$. This implies that $\|f\|_{N_2} \leq \sqrt{2e}\|a\|_2$ by the definition of the norm in the Orlicz space. Similarly, we obtain the following inequality for partial sums $S_n = \sum_{k=1}^{n} a_k r_k$ of series (3.2):

$$\|S_m - S_n\|_{N_2} \leq \sqrt{2e} \left(\sum_{k=n+1}^{m} a_k^2 \right)^{1/2}, \quad 1 \leq n < m < \infty.$$

Then, due to (3.3), $S_n \to f$ L_{N_2} as $n \to \infty$. Therefore, $f \in G$ and the theorem is proved. □

Theorem 3.1 holds if the space G is replaced by an arbitrary symmetric space X such that $G \subset X$. At the same time, the inequality

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \leq C\|a\|_2 \tag{3.4}$$

holds with a positive constant C depending only on X. Indeed, similarly to the proof of the completeness of E_X at the beginning of the chapter, we can show that the linear operator

$$Ra(t) = \sum_{k=1}^{\infty} a_k r_k \quad (0 \leq t \leq 1)$$

mapping ℓ_2 to X has a closed graph and, therefore, is bounded. Formulate the result as follows.

Proposition 3.1. *If the symmetric space $X \supset G$, then the subspace $\mathcal{R}(X)$ coincides with the closed linear span $[r_n]_X$.*

Proof. The embedding $[r_n]_X \subset \mathcal{R}(X)$ is a direct corollary of the definitions.

Now, assume that

$$f = \sum_{k=1}^{\infty} a_k r_k \in \mathcal{R}(X).$$

Since, in particular, this series converges a.e., $a = (a_k)_{k=1}^{\infty} \in \ell_2$ by Theorem 1.2. Then relation (3.4) holds. Therefore, the corresponding series converges in X, i.e., $f \in [r_n]_X$. □

Then, due to the embedding $X \subset L_1$, which holds for any symmetric space X on $[0,1]$ (see [92, Theorem 2.4.1]), and Theorem 1.4, we have the inequality

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \geq c \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_1 \geq \frac{c}{2}\|a\|_2, \tag{3.5}$$

i.e., the inequality inverse to (3.4) holds always. Finally, the embedding $G \subset X$ ensures that

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \asymp \|a\|_2 \tag{3.6}$$

with constants depending only on X. From the geometrical point of view, relation (3.6) means that the Rademacher system in symmetric space X is equivalent to a canonical basis in ℓ_2, i.e. $E_X = \ell_2$. It turns out that the inverse assertion holds as well.

Theorem 3.2. *If X is a symmetric space on $[0,1]$, then relation (3.6) holds if and only if $G \subset X$.*

We begin with several lemmas. In the first, as usually, $n_x(z) = m\{\omega : x(\omega) > z\}$ is the distribution function for a function $x(\omega)$ measurable on I or $I \times I$.

Lemma 3.1. *If $n_{|x_k|}(z) \to n_{|x|}(z)$ as $k \to \infty$ for any $z > 0$, then $x_k^*(t) \to x^*(t)$ $(k \to \infty)$ at all points of continuity of a rearrangement $x^*(t)$ (and, therefore, a.e.).*

Proof. Let $x^*(t)$ be continuous at the point t_0 and $\varepsilon > 0$. For $z_1 = x^*(t_0) + \varepsilon$, we have $n_{|x|}(z_1) < t_0$ and, therefore, $n_{|x_k|}(z_1) < t_0$ if $k \geq k_1$. Then, by the definition of rearrangement,

$$x_k^*(t_0) \leq x^*(t_0) + \varepsilon \ \ (k \geq k_1). \tag{3.7}$$

On the other hand, due to the continuity $x^*(t)$ at the point t_0, there exists $\delta > 0$ such that

$$x^*(t_0 + \delta) \geq x^*(t_0) - \varepsilon. \tag{3.8}$$

If $z_2 = x^*(t_0 + \delta) - \varepsilon$, then $n_{|x|}(z_2) \geq t_0 + \delta$ and, therefore, there exists k_2 such that $n_{|x_k|}(z_2) \geq t_0$ as $k \geq k_2$. Therefore, due to (3.8), we have

$$x_k^*(t_0) \geq x^*(t_0) - 2\varepsilon \ \ (k \geq k_2).$$

For $k \geq \max(k_1, k_2)$, the previous inequality and inequality (3.7) hold. Combining them, we obtain the assertion of the lemma. □

In the following lemma we modify the definition of Orlicz spaces. Let F be an increasing function on $(0, \infty)$ such that $F(0) = 0$. Assume that the function $e^{F(t)} - 1$ is convex for sufficiently large $t > 0$. Then it generates the Orlicz space on $[0,1]$ denoted by $\operatorname{Exp} L^F$.

Lemma 3.2. *Let a function $F(t)$ be convex for $t > 0$ or $F(t) = t^\alpha$ for some $\alpha > 0$. Then the space $\operatorname{Exp} L^F$ coincides with the Marcinkiewicz space $M(\psi_F)$, $\psi_F(t) = tF^{-1}(\ln(e/t))$ (F^{-1} is the function inverse to F) and the following relation holds (with constants depending only on F):*

$$\|x\|_{\operatorname{Exp} L^F} \asymp \sup_{0<s\leq 1} \frac{x^*(s)}{F^{-1}(\ln(e/s))}.$$

In particular, the Orlicz space L_{N_α}, $N_\alpha(t) = \exp(t^\alpha) - 1$ $(\alpha > 0)$ coincides with the Marcinkiewicz space $M(\psi_\alpha)$, $\psi_\alpha(t) = t \ln^{1/\alpha}(e/t)$. With constants depending only on α, the following expression holds:

$$\|x\|_{N_\alpha} \asymp \sup_{0<s\leq 1} \bigl(x^*(s) \ln^{-1/\alpha}(e/s)\bigr).$$

Proof. Denote by $M(\psi_F)_w$ a functional space with the quasinorm

$$\|x\|_{M(\psi_F)_w} = \sup_{0<s\leq 1} \frac{x^*(s)}{F^{-1}(\ln(e/s))}.$$

Due to the inequality $x^*(t) \leq 1/t \int\limits_0^t x^*(s)\,ds$ and the definition of the norm in the Marcinkiewicz space, we have

$$M(\psi_F) \subset M(\psi_F)_w \quad \text{and} \quad \|x\|_{M(\psi_F)_w} \leq \|x\|_{M(\psi_F)} \ \ (x \in M(\psi_F)). \tag{3.9}$$

At the same time, using the expression for fundamental functions of Orlicz spaces (see Sec. 0.2), we obtain

$$\varphi_{\operatorname{Exp} L^F}(t) = 1/F^{-1}(\ln(1 + 1/t)) \asymp 1/F^{-1}(\ln(e/t)).$$

Therefore, since the Marcinkiewicz space is maximal among all symmetric spaces with the same fundamental function (see [92, Theorem 2.5.7]), we have

$$\operatorname{Exp} L^F \subset M(\psi_F) \quad \text{and} \quad \|x\|_{M(\psi_F)} \le \|x\|_{\operatorname{Exp} L^F} \ \ (x \in \operatorname{Exp} L^F). \tag{3.10}$$

Then one can easily check that the function $x_0(t) := F^{-1}(\ln(e/t))$ belongs to the Orlicz space $\operatorname{Exp} L^F$. Moreover, by the definition of space $M(\psi_F)_w$, the following inequality holds:

$$x^*(t) \le \|x\|_{M(\psi_F)_w} x_0(t) \ \ (x \in M(\psi_F)_w)$$

(i.e., $x_0(t)$ is the function with the maximal rearrangement in the space $M(\psi_F)_w$). Therefore,

$$M(\psi_F)_w \subset \operatorname{Exp} L^F \quad \text{and} \quad \|x\|_{\operatorname{Exp} L^F} \le \|x_0\|_{\operatorname{Exp} L^F} \, \|x\|_{M(\psi_F)_w} \ \ (x \in M(\psi_F)_w).$$

Combining the last expressions with (3.9) and (3.10), we obtain both statements of the lemma. □

In the last lemma, we consider a separable part of the second associate space X''. Recall (for details, see Sec. 0.1) that (the first) associated space X' to symmetric space X consists of all measurable on $[0,1]$ functions $y = y(t)$ such that

$$\|y\|_{X'} = \sup\left\{ \int_0^1 x(t)y(t)\,dt : \, x \in X, \|x\|_X \le 1 \right\} < \infty.$$

Lemma 3.3. *For any symmetric space X on $[0,1]$, the isometric relation $(X'')^\circ = X^\circ$ holds.*

Proof. The statement of the lemma is obvious for $X = L_\infty$. Assume that

$$X \ne L_\infty. \tag{3.11}$$

Then, for the fundamental function of X, we have

$$\lim_{s\to 0} \varphi_X(s) = 0. \tag{3.12}$$

Indeed, if we assume that $\varphi_X(s) \ge c > 0$ ($s \in (0,1]$) for some $c > 0$, then, due to the symmetry of X, we have

$$\|x\|_X = \|x^*\|_X \ge \|x^* \chi_{(0,s)}\|_X \ge x^*(s)\varphi_X(s) \ge c x^*(s) \ \ (0 < s \le 1),$$

whence $\|x\|_\infty \le c^{-1}\|x\|_X$ ($x \in X$) and, therefore, $X \subset L_\infty$. Since the inverse embedding $L_\infty \subset X$ holds always (see [92, Theorem 2.4.1]), we arrive at a contradiction to assumption (3.11).

Now we show that the space X° is separable under Condition (3.11). Let $x \in X^\circ$ and $\varepsilon > 0$. First of all, there exists $y \in L_\infty$ such that $\|x - y\|_X < \varepsilon/3$. Assume that $\|y\|_\infty = M$ and, using (3.12), choose $h > 0$ such that $\varphi_X(h) < \varepsilon/(6M)$. By the Luzin theorem (see [110, Theorem 4.4.4]), there exists a continuous on $[0,1]$ function z such that $\|z\|_\infty \le M$ and $m\{t \in [0,1] : z(t) \ne y(t)\} < h$. Then

$$\|y - z\|_X \le \|(y-z)\chi_{\{y\ne z\}}\|_X \le 2M\varphi_X(h) < \frac{\varepsilon}{3}.$$

Finally, using the uniform continuity of the function z, we can find a function

$$u(t) = \sum_{k=1}^{2^n} c_k \chi_{[(k-1)2^{-n}, k2^{-n})}, \tag{3.13}$$

where $n \in \mathbb{N}$ and c_k are rational. This function is such that $\|z - u\|_X < \varepsilon/3$. Since, from the given inequalities, it follows that $\|x - u\|_X < \varepsilon$ and the set of functions of form (3.13) is countable, it follows that the separability of X° is obtained.

Since X° is separable, it follows that it is isometrically embedded into $(X^\circ)''$ (see [84, Theorem 4.3.3 and Corollary 6.1.1]). On the other hand, from the definition of X' and properties of integrals, it follows that $(X^\circ)' = X'$ and, therefore, $(X^\circ)'' = X''$. Therefore, X° is isometrically embedded into the second associate space X''. In particular, this implies that norms from X and X'' are equal to each other on L_∞. Finally, $(X'')^\circ = X^\circ$. □

Proof of Theorem 3.2. By Theorem 3.1 and remarks after it, we need to prove only the following: if X is an arbitrary symmetric space and

$$\left\|\sum_{k=1}^{\infty} a_k r_k\right\|_X \le C\|a\|_2, \tag{3.14}$$

then $G \subset X$.

The idea of the proof is to apply the central limit theorem. It is formulated in Sec. 0.3 (for details, see [46, Theorem 7.2]). Apply this theorem to the function sequence

$$v_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} r_k(t), \qquad n = 1, 2, \ldots,$$

defined on the segment $[0,1]$ considered, as usually, with the Lebesgue measure. Due to the symmetry of the distribution $v_n(t)$, we have

$$\lim_{n\to\infty} n_{|v_n|}(\tau) = \lim_{n\to\infty} m\{t \in [0,1] : |v_n(t)| > \tau\} = \Psi(\tau) \ \ (\tau > 0),$$

where

$$\Psi(\tau) = \frac{2}{\sqrt{2\pi}} \int_{\tau}^{\infty} e^{-t^2/2}\, dt.$$

Then, by Lemma 3.1, we have

$$\lim_{n\to\infty} v_n^*(s) = \Psi^{-1}(s) \ \ (0 < s \le 1)$$

($\Psi^{-1}(s)$ is the function inverse to $\Psi(\tau)$). Using (3.14), we have $\|v_n\|_X \le C$ $(n = 1, 2, \ldots)$ and, therefore, $\Psi^{-1}(s) \in X''$ due to the maximality of the space X''. The estimate

$$\int_{\tau}^{\infty} e^{-\frac{t^2}{2}}\, dt \ge \int_{\tau}^{2\tau} e^{-\frac{t^2}{2}}\, dt \ge e^{-2\tau^2}, \quad \tau \ge 1,$$

shows that $\Psi(\tau) \ge 2/\sqrt{2\pi} e^{-2\tau^2}$ $(\tau \ge 1)$ and, since X'' is symmetric, it follows that $\ln^{1/2}(e/t) \in X''$. Therefore, $L_{N_2} \subset X''$ by Lemma 3.2. Using lemma 3.3, we obtain $G \subset (X'')^\circ = X^\circ \subset X$. The theorem is proved. ☐

Remark 3.1. From the above proof of the theorem, it follows that the embedding $G \subset X$ is equivalent to the following condition, which is weaker than (3.6):

$$\sup_{n\in\mathbb{N}} \frac{1}{\sqrt{n}} \Big\|\sum_{k=1}^{n} r_k\Big\|_X < \infty.$$

Consider the problem to find a symmetric space in which the Rademacher system is equivalent to a canonical basis in ℓ_1, i.e., $E_X = \ell_1$. First of all, due to Remark 1.5, this holds for the space L_∞. It turns out that no other symmetric space with such property exists (up to the norm equivalence).

Theorem 3.3. *If X is symmetric space on* $[0,1]$, *then*

$$\left\|\sum_{k=1}^{\infty} a_k r_k\right\|_X \asymp \|a\|_1$$

if and only if $X = L_\infty$.

Begin with the following simple statement.

Lemma 3.4. *Let X be a symmetric space on $[0,1]$. Then $X = L_\infty$ (with equivalence of norms) if and only if there exists a constant $c > 0$ and a sequence $\{x_n(t)\} \subset X$ such that $|x_n(t)| \le 1$, $x_n(t) \to 0$ a.e. on $[0,1]$, and $\|x_n\|_X \ge c$.*

Proof. First of all, if $X = L_\infty$, then the sequence $x_n(t) = \chi_{(0,1/n)}(t)$ has all the properties mentioned in lemma.

Assume that $|x_n(t)| \le 1$ and $x_n(t) \to 0$ for almost every $t \in [0,1]$. Since the convergence a.e. implies the convergence in measure, the following inequality holds for every $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$:

$$m\left\{t \in [0,1] : |x_n(t)| \ge \frac{c}{2\|\chi_{[0,1]}\|_X}\right\} < \varepsilon.$$

Denote

$$E = \left\{t \in [0,1] : |x_n(t)| \ge \frac{c}{2\|\chi_{[0,1]}\|_X}\right\} \quad \text{and} \quad E' = [0,1] \setminus E.$$

Then $m(E) < \varepsilon$ and, by the condition, we have

$$\|\chi_E\|_X \ge \|x_n\chi_E\|_X \ge \|x_n\|_X - \|x_n\chi_{E'}\|_X \ge c - \frac{c}{2\|\chi_{[0,1]}\|_X}\|\chi_{E'}\|_X \ge \frac{c}{2}.$$

Since $\varepsilon > 0$ is arbitrary we have $\varphi_X(s) \ge \dfrac{c}{2} > 0$ for all $s \in (0,1]$. As in the proof of Lemma 3.3, one can obtain that $X \subset L_\infty$. Since the inverse embedding holds always, it follows that the lemma is proved. $\square$

Proof of Theorem 3.3. It suffices to prove that the relation $X = L_\infty$ follows from the inequality

$$\left\|\sum_{k=1}^{\infty} a_k r_k\right\|_X \ge c\|a\|_1 \tag{3.15}$$

Consider the sequence of functions

$$x_n(t) = \frac{1}{n}\sum_{k=1}^{n} r_k(t), \quad n = 1, 2, \ldots$$

We have $|x_n(t)| \le 1$ and $\|x_n\|_X \ge c$ due to (3.15). The fact that $x_n(t) \to 0$ a.e. on $[0,1]$ follows from the strong law of large averages (see [46, Theorem 9.2.5]). Therefore, $X = L_\infty$ by Lemma 3.4 (with the norm equivalence). $\square$

In conclusion, we consider the complementability problem for $\mathcal{R}(X)$. Recall that a linear operator P acting in a linear space E is called a *projector* if $P^2 = P$. A closed linear subspace H of a linear normed space X is called *complemented* if there exists a projector P bounded in X and its direct image coincides with H.

In the next theorem, the associate space G' is used together with G. To describe it, recall that, by Lemma 3.2, the Orlicz space L_{N_2} coincides with the Marcinkiewicz space $M(\psi_2)$, where $\psi_2(t) = t\ln^{1/2}(e/t)$. Then $G = M(\psi_2)^\circ$. Therefore, G' is the Lorentz space $\Lambda(\psi_2)$ (see [92, Theorem 2.5.4]). Calculations show that $(\psi_2(t))' \asymp \ln^{1/2}(e/t)$ $(0 < t \le 1)$. That is why the space G' consists of all measurable functions $x(t)$ such that

$$\int_0^1 x^*(t)\ln^{1/2}\frac{e}{t}\,dt < \infty \quad \text{and} \quad \|x\|_{G'} \asymp \int_0^1 x^*(t)\ln^{1/2}\frac{e}{t}\,dt.$$

Theorem 3.4. *The subspace $\mathcal{R}(X)$ is complemented in a symmetric space X on $[0,1]$ if and only if*

$$G \subset X \subset G'. \tag{3.16}$$

Proof. First, assume that condition (3.16) holds. Then $G'' \subset X'$ and $G \subset X'$. Therefore, by Theorem 3.2, the Rademacher system is equivalent to the canonical basis in ℓ_2, both in X and X'. This means that there exists $A > 0$ such that

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X \le A\|a\|_2 \quad \text{and} \quad \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X'} \le A\|a\|_2. \tag{3.17}$$

Show that the orthogonal projector

$$Pf(t) := \sum_{k=1}^{\infty} \int_0^1 f(u) r_k(u)\, du \cdot r_k(t) \tag{3.18}$$

is bounded in X. Indeed, if $c_k(f) := \int_0^1 f(u) r_k(u)\, du$, then, in view of (3.17), for any $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} c_k(f)^2 = \int_0^1 f(u) \sum_{k=1}^{n} c_k(f) r_k(u)\, du \le \|f\|_X \Big\|\sum_{k=1}^{n} c_k(f) r_k\Big\|_{X'} \le A\|f\|_X \left(\sum_{k=1}^{n} c_k(f)^2\right)^{1/2},$$

and, therefore, the following inequality holds:

$$\Big(\sum_{k=1}^{\infty} c_k(f)^2\Big)^{1/2} \le A\|f\|_X \ \ (f \in X).$$

This and (3.17) implies that

$$\|Pf\|_X \ \le A\Big(\sum_{k=1}^{\infty} c_k(f)^2\Big)^{1/2} \le A^2\|f\|_X$$

for every $f \in X$. Therefore, the subspace $\mathcal{R}(X)$ is complemented in X.

Prove the inverse statement. First, show that it suffices to prove the boundedness of the orthogonal projector P defined by (3.18) in X. First of all (see the reasoning below the proof of Theorem 3.1), we have

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X \ge c\|a\|_2 \quad \text{and} \quad \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X'} \ge c\|a\|_2,$$

where $c > 0$ is a universal constant. Therefore, if $\|Pf\|_X \le B\|f\|_X$, then, for arbitrary $f \in X$, $n \in \mathbb{N}$, and $a_k \in \mathbb{R}$ $(k = 1, 2, \ldots, n)$ we have

$$\begin{aligned}\int_0^1 f(t) \sum_{k=1}^{n} a_k r_k(t)\, dt &= \sum_{k=1}^{n} a_k c_k(f) \le \|a\|_2 \Big(\sum_{k=1}^{n} c_k(f)^2\Big)^{1/2} \\ &\le c^{-1}\|a\|_2\|Pf\|_X \le c^{-1}B\|a\|_2\|f\|_X.\end{aligned}$$

This implies the inequality

$$\Big\|\sum_{k=1}^{n} a_k r_k\Big\|_{X'} \le c^{-1}B\|a\|_2 \ \ (n \in \mathbb{N}).$$

If $a = (a_k)_{k=1}^{\infty} \in \ell_2$, then a similar inequality for the sum $\sum_{k=n}^{m} a_k r_k$ $(n < m)$ implies the convergence of the series $\sum_{k=1}^{\infty} a_k r_k$ in the space X' and the inequality

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X'} \le c^{-1}B\|a\|_2. \tag{3.19}$$

Further, if $f \in X$ and $g \in X'$, then for any $n \in \mathbb{N}$, we have

$$\int_0^1 f(u) \sum_{k=1}^n c_k(g) r_k(u)\, du = \int_0^1 g(t) \sum_{k=1}^n c_k(f) r_k(t)\, dt \le \|Pf\|_X \|g\|_{X'} \le B\|f\|_X \|g\|_{X'},$$

whence

$$\Big\| \sum_{k=1}^n c_k(g) r_k \Big\|_{X'} \le B\|g\|_{X'} \quad (n \in \mathbb{N}).$$

Since the space X' is maximal, it follows from the last inequality that the projector P is bounded in X' with the norm not greater than B. Repeating the previous reasoning, we obtain the inequality

$$\Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{X''} \le c^{-1} B \|a\|_2. \tag{3.20}$$

Applying Theorem 3.2 to the spaces X' and X'', we come to the conclusion that it follows from (3.19) and (3.20) that $G \subset X'$ and $G \subset X''$. From the first embedding, we obtain that $X \subset X'' \subset G'$; and from the second one, we see that $G \subset (X'')^\circ$. Since $(X'')^\circ = X^\circ$ by Lemma 3.3, it follows from the last embedding that $G \subset X^\circ \subset X$. Finally, we obtain (3.16). Therefore, it remains to show that the boundedness of the projector P follows from the complementability of $\mathcal{R}(X)$ in X

Let $t = \sum\limits_{i=1}^\infty \alpha_i 2^{-i}$ and $u = \sum\limits_{i=1}^\infty \beta_i 2^{-i}$ $(\alpha_i, \beta_i = 0, 1)$ be the binary expansion of the numbers $t, u \in [0,1]$. Define the following operation:

$$t \oplus u = \sum_{i=1}^\infty 2^{-i} [(\alpha_i + \beta_i) \bmod 2].$$

One can easily verify that this operation transforms the segment $[0,1]$ into a compact Abelian group. For every $u \in [0,1]$, the transformation $w_u(s) = s \oplus u$ preserves the Lebesgue measure on $[0,1]$, i.e., for any measurable $E \subset [0,1]$, its inverse image $w_u^{-1}(E)$ is measurable and $m(w_u^{-1}(E)) = m(E)$. Then the operators $T_u f = f \circ w_u$ $(0 \le u \le 1)$ isometrically act in X. It follows from the definition of the Rademacher functions that the subspace $\mathcal{R}(X)$ is invariant with respect to these operators. Therefore, by the Rudin theorem (see [127, Theorem 1.5.18]), there exists a bounded operator Q acting from X into $\mathcal{R}(X)$ and commuting with all operators T_u $(0 \le u \le 1)$. Show that $Q = P$.

First of all, represent the projector Q as follows:

$$Qf(t) = \sum_{i=1}^\infty Q_i(f) r_i(t), \tag{3.21}$$

where Q_i $(i = 1, 2, \ldots)$ are linear bounded functionals on X by Corollary 1.7. It is obvious that

$$Q_i(r_j) = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases} \tag{3.22}$$

Consider the sets

$$\gamma_i = \Big\{ u \in [0,1] : u = \sum_{j=1}^\infty \alpha_j 2^{-j}, \ \ \alpha_i = 0 \Big\} \quad , \quad \overline{\gamma_i} = [0,1] \backslash \gamma_i.$$

One can check that

$$r_i(t \oplus u) = \begin{cases} r_i(t), & u \in \gamma_i, \\ -r_i(t), & u \in \overline{\gamma_i}. \end{cases}$$

Due to the relation $T_u Q = QT_u$ $(0 \le u \le 1)$, this implies

$$Q_i(T_u f) = \begin{cases} Q_i(f), & u \in \gamma_i, \\ -Q_i(f), & u \in \overline{\gamma_i}. \end{cases}$$

Taking into account that $m(\gamma_i) = m(\overline{\gamma_i}) = 1/2$, we find that

$$\int\limits_{\gamma_i} Q_i(T_u f)\, du = \frac{1}{2} Q_i(f) \quad \text{and} \quad \int\limits_{\overline{\gamma_i}} Q_i(T_u f)\, du \; = \; -\frac{1}{2} Q_i(f).$$

Due to the boundedness of Q_i, this functional can be moved outside the integral; therefore, we obtain

$$Q_i(f) = Q_i \Biggl(\int\limits_{\gamma_i} T_u f\, du - \int\limits_{\overline{\gamma_i}} T_u f\, du \Biggr). \tag{3.23}$$

Since

$$\{s \in [0,1] : \; s = t \oplus u, \;\; u \in \gamma_i\} = \begin{cases} \gamma_i, & t \in \gamma_i, \\ \overline{\gamma_i}, & t \in \overline{\gamma_i}, \end{cases}$$

and

$$\{s \in [0,1] : \; s = t \oplus u, \;\; u \in \overline{\gamma_i}\} = \begin{cases} \overline{\gamma_i}, & t \in \gamma_i, \\ \gamma_i, & t \in \overline{\gamma_i}, \end{cases}$$

and the transformation ω_u preserves the Lebesgue measure on $[0,1]$, we have

$$\int\limits_{\gamma_i} T_u f(t) du = \int\limits_{\gamma_i} f(s)\, ds\, \chi_{\gamma_i}(t) + \int\limits_{\overline{\gamma_i}} f(s)\, ds\, \chi_{\overline{\gamma_i}}(t)$$

and

$$\int\limits_{\overline{\gamma_i}} T_u f(t)\, du = \int\limits_{\overline{\gamma_i}} f(s)\, ds\, \chi_{\gamma_i}(t) + \int\limits_{\gamma_i} f(s)\, ds\, \chi_{\overline{\gamma_i}}(t).$$

One can check that $r_i(t) = \chi_{\gamma_i}(t) - \chi_{\overline{\gamma_i}}(t)$. Therefore, from the last two relations, it follows that

$$\int\limits_{\gamma_i} T_u f(t)\, du - \int\limits_{\overline{\gamma_i}} T_u f(t)\, du = \int\limits_0^1 f(s) r_i(s)\, ds\, r_i(t).$$

This and (3.21)–(3.23) yield

$$Q_i(f) = \int\limits_0^1 f(s) r_i(s)\, ds,$$

i.e., $Q = P$. The theorem is proved. ☐

Comments and References. This chapter is devoted to a detailed study of the properties of the Rademacher system in general symmetric space. Theorem 3.1 has been known for a long time ago (see e.g., [142, Theorem 5.8.7]). The main result of this chapter is Theorem 3.2, which is proved by Rodin and Semenov in [124]. It shows how "far" from L_∞ a symmetric space must be in order for the Rademacher system to "span" a coordinate space ℓ_2 in it. This property is very important for studying different questions related to the geometry of symmetric spaces. For example, in [74] this property is used to prove that a canonical embedding $X \subset L_1$ is not strictly singular if and only if $G \subset X$ (see also [29]).

An important role in the future is played by Lemma 3.2 on the coincidence of the exponential Orlicz space and the Marcinkiewicz space (the characterization of such spaces is given in [99, 128]). Theorem 3.3 is proved in [124] as well. Theorem 3.4 was proved independently by Rodin and Semenov in [125] and Lindenshtraus and Tsafriri (see [97, Theorem 2.b.4(ii)]); our proof follows the scheme given in [125]. Theorems 3.2 and 3.4 are the source of many interesting investigations. Let us mention just several (others will be considered in the following chapters). In [48], it is shown that if the sequence $\{f_k\}_{k=1}^{\infty}$ of independent identically distributed random variables "spans" ℓ_2 in symmetric space X, then $f_1 \in L_2$ and $G \subset X$. Moreover, if the subspace generated by $\{f_k\}_{k=1}^{\infty}$ is complemented in X, then $G \subset X \subset G'$. More general problems related to Semenov's question were considered by Raynaud in [122]. Assume that a separable symmetric space X does not contain ℓ_2 as a sublattice. Then the condition $G \subset X$ is necessary and sufficient for X to contain ℓ_2 as a subspace. If a separable symmetric space X does not contain ℓ_2 as a complementary sublattice, then the condition $G \subset X \subset G'$ is necessary and sufficient for X to contain ℓ_2 as a complementary subspace. Similar (but not so general) results were obtained by Tokarev in [136].

Another direction of investigations is related to the changing of scalar multiple Rademacher functions (or functions of any other fixed sequence) in (3.14) by functions of systems from some class. As was proved in [79], for any $C > 0$ depending only on symmetric space X and all sequences of martingale differences $\{f_k\}_{k=1}^{\infty} \subset X$ the inequality

$$\Big\|\sum_{k=1}^{\infty} f_k\Big\|_X \le C\Big\|\Big(\sum_{k=1}^{\infty} f_k^2\Big)^{1/2}\Big\|_X \tag{3.24}$$

holds if and only if $\alpha_X > 0$ (α_X is the lower Boyd index of the space X). If we consider (instead of all martingale differences) only sequences of mean zero independent functions $\{f_k\}_{k=1}^{\infty} \subset X$, i.e., assume that $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \ldots)$, then inequality (3.24) holds if and only if the space X has the so-called Kruglov property (see [25] and [33–35, 49]). Note that from the condition $\alpha_X > 0$, it follows that X has the Kruglov property (see [49, Theorem 1.3, p. 16]). At the same time, if X has the Kruglov property, then $G \subset X$.

Chapter 4

RADEMACHER SYSTEMS IN SYMMETRIC SPACES "CLOSE" TO L_∞

In the previous chapter it was shown that in the case where symmetric space X contains the separable part G of the Orlicz space L_{N_2}, $N_2(t) = e^{t^2} - 1$, the Rademacher system is equivalent in X to the canonical basis in ℓ_2. It turns out that the behavior of this system in spaces more "close" to L_∞ is more varied. To make these words more exact, we formulate the theorems from the previous chapter, using the same notation: $E_X = \ell_2$ if and only if $X \supset G$ (Theorem 3.2) and $E_X = \ell_1$ if and only if $X = L_\infty$ (Theorem 3.3). What sequence spaces could be obtained a similar way? The answer is as follows: The class of such spaces exactly coincides with the set of interpolation spaces with respect to the couple (ℓ_1, ℓ_2). At the same time, there exists a one-to-one correspondence between them and symmetric function interpolation spaces with respect to the couple (L_∞, G). The theory of interpolation of linear operators (see Sec. 0.5) plays the main role in studying these questions.

To formulate the first theorem, we need the following definition.

Definition 4.1. Let (X_0, X_1) be a Banach couple, Y_0 be a closed subspace of X_0, and Y_1 be a closed subspace of X_1. Then the Banach couple (Y_0, Y_1) is called a $\mathcal{K}$-*subcouple* ($\mathcal{K}$-*closed subcouple*) of the couple (X_0, X_1) if

$$\mathcal{K}(t, y; Y_0, Y_1) \asymp \mathcal{K}(t, y; X_0, X_1)$$

with constants not depending on $y \in Y_0 + Y_1$ and $t > 0$.

Define the operator R as follows: if a sequence of reals $a = (a_k)_{k=1}^\infty$ belongs to ℓ_2, then

$$Ra(t) = \sum_{k=1}^{\infty} a_k r_k(t), \quad t \in [0,1]. \tag{4.1}$$

It follows from Theorem 1.1 that, for fixed $a \in \ell_2$, relation (4.1) defines a measurable finite a.e. function. Moreover, by Theorem 3.1, R is a linear bounded operator from ℓ_2 to G and

$$\|Ra\|_G \asymp \|a\|_{\ell_2}; \tag{4.2}$$

due to Remark 1.5, this operator acts isometrically from ℓ_1 to L_∞, i.e.,

$$\|Ra\|_\infty = \|a\|_{\ell_1}. \tag{4.3}$$

Therefore, $(R(\ell_1), R(\ell_2))$ can be considered as a subcouple of the Banach couple (L_∞, G) ($R(\ell_1)$ is a subspace of L_∞ and $R(\ell_2)$ is a subspace of G).

From relations (4.2) and (4.3) and from the definition of the $\mathcal{K}$-functional, we obtain that

$$\mathcal{K}(t, Ra; R(\ell_1), R(\ell_2)) \asymp \mathcal{K}(t, a; \ell_1, \ell_2). \tag{4.4}$$

Theorem 4.1. *The Banach couple $(R(\ell_1), R(\ell_2))$ is a $\mathcal{K}$-subcouple of the Banach couple (L_∞, G), i.e., the following relation holds:*

$$\mathcal{K}(t, Ra; L_\infty, G) \asymp \mathcal{K}(t, a; \ell_1, \ell_2) \tag{4.5}$$

with constants not depending on $a = (a_k)_{k=1}^\infty \in \ell_2$ and $t > 0$.

Proof. As known from [59], the $\mathcal{K}$-functional in a couple of Marcinkiewicz spaces can be found (accurate within equivalence) as follows:

$$\mathcal{K}(t, x; M(\varphi_0), M(\varphi_1)) \asymp \sup_{0<u\le 1} \frac{\int_0^u x^*(s)\,ds}{\max(\varphi_0(u), \varphi_1(u)/t)}, \quad t > 0. \tag{4.6}$$

By Lemma 3.2, the Orlicz space L_{N_2} coincides with the Marcinkiewicz space $M(\varphi_1)$ constructed by the function $\varphi_1(u) = u \cdot \ln^{1/2}(e/u)$. Since $L_\infty = M(\varphi_0)$, where $\varphi_0(u) = u$, it follows from the previous relation that for $x \in G$,

$$\mathcal{K}(t, x; L_\infty, G) = \mathcal{K}(t, x; L_\infty, L_{N_2}) \asymp \sup_{0<u\le 1} \left\{ \frac{1}{u} \int_0^u x^*(s)\,ds \, \min(1, t \cdot \ln^{-1/2}(e/u)) \right\}. \tag{4.7}$$

Since $x^*(u) \le 1/u \int_0^u x^*(s)\,ds$, it follows that

$$\mathcal{K}(t, x; L_\infty, G) \ge c \sup_{0<u\le 1} \left\{ x^*(u) \min(1, t \cdot \ln^{-1/2}(e/u)) \right\}. \tag{4.8}$$

Let $a = (a_k)_{k=1}^\infty \in \ell_2$. By Corollary 2.2 (see (2.20)), there exist universal constants $\beta \in (0,1)$ and $C \ge 1$ such that

$$(Ra)^*(\beta u) \ \ge C^{-1} \mathcal{K}(\ln^{1/2}(e/u), a; \ell_1, \ell_2) \ \ (0 < u \le 1),$$

i.e.,

$$(Ra)^*(\beta e^{1-t^2}) \ge C^{-1} \mathcal{K}(t, a; \ell_1, \ell_2) \ \ (t \ge 1). \tag{4.9}$$

On the other hand, due to (4.8), we have

$$\begin{aligned} \mathcal{K}(t, Ra; L_\infty, G) &\ge c (Ra)^*(\beta e^{1-t^2}) \min(1, t \cdot \ln^{-1/2}(\beta^{-1} e^{t^2})) \\ &\ge c (Ra)^*(\beta e^{1-t^2}) \min\left(1, \frac{t}{\sqrt{t^2 - \ln \beta}}\right). \end{aligned}$$

Since the function $t/\sqrt{t^2 - \ln\beta}$ $(t > 0)$ increases, it follows that for $t \geq 1$, we have

$$\mathcal{K}(t, Ra; L_\infty, G) \geq C_1 (Ra)^*(\beta e^{1-t^2}),$$

where $C_1 = c/\sqrt{1 - \ln\beta}$. Therefore, due to inequality (4.9), we have

$$\mathcal{K}(t, Ra; L_\infty, G) \geq \frac{C_1}{C} \mathcal{K}(t, a; \ell_1, \ell_2) \ \ (t \geq 1).$$

If $0 < t \leq 1$, then using the inequalities $\|a\|_2 \leq \|a\|_1$ $(a \in \ell_1)$ and $\|x\|_G \leq \|x\|_\infty$ $(x \in L_\infty)$, the definition of $\mathcal{K}$-functional, and relation (4.2), we obtain

$$\mathcal{K}(t, Ra; L_\infty, G) \ = \ t\|Ra\|_G \ \geq \ C_2 t \|a\|_2 \ = \ C_2 \mathcal{K}(t, a; \ell_1, \ell_2).$$

Therefore, the inequality

$$\mathcal{K}(t, Ra; L_\infty, G) \geq \min\Big(C_2, \frac{C_1}{C}\Big) \mathcal{K}(t, a; \ell_1, \ell_2)$$

holds for all $t > 0$. The inverse inequality

$$\mathcal{K}(t, Ra; L_\infty, G) \leq C_3 \mathcal{K}(t, a; \ell_1, \ell_2) \ \ (t > 0)$$

follows from (4.4) and the definition of the $\mathcal{K}$-functional. □

The proved statement allows us to establish a one-to-one correspondence between symmetric function interpolation spaces with respect to the couple (L_∞, G) and the interpolation spaces of sequences with respect to the couple (ℓ_1, ℓ_2). To formulate the result, we need a notion of the real interpolation $\mathcal{K}$-method (see Sec. 0.5): its terms allow us to describe all interpolation spaces with respect to any $\mathcal{K}$-monotone couple (see [50, Theorem 15.1]). In particular, by the Sparr theorem (see [131]), the couple (ℓ_1, ℓ_2) is $\mathcal{K}$-monotone; therefore, if $E \in \mathcal{I}(\ell_1, \ell_2)$, then

$$E = (\ell_1, \ell_2)^{\mathcal{K}}_F \tag{4.10}$$

for a certain parameter of the real $\mathcal{K}$-method F. Using that and Theorem 4.1, for a given coordinate interpolation space E between ℓ_1 and ℓ_2 we can find a symmetric function space X such that a Rademacher series belongs to X if and only if the corresponding sequence of coefficients belongs to E.

Corollary 4.1. *Let $E \in \mathcal{I}(\ell_1, \ell_2)$ and let it be defined by relation (4.10). Then for the symmetric space $X = (L_\infty, G)^{\mathcal{K}}_F$, the relation*

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X \asymp \|a\|_E$$

holds with constants not depending on the sequence $a = (a_k)_{k=1}^\infty$.

Proof. It suffices to apply Theorem 4.1 and the definitions of the spaces of the real $\mathcal{K}$-method. □

Now we can give a description of all sequence spaces E_X induced by the Rademacher system means of (3.1).

Corollary 4.2. *A Banach sequence space E coincides with a space of the form E_X (with the equivalence of the norms) if and only if $E \in \mathcal{I}(\ell_1, \ell_2)$.*

Proof. If $E \in \mathcal{I}(\ell_1, \ell_2)$, then there exists F such that relation (4.10) holds. Due to the previous corollary, we have $E = E_X$, where $X := (L_\infty, G)^{\mathcal{K}}_F$.

Suppose, to the contrary, that $E = E_X$ for some symmetric space X. Let U be a linear operator bounded in ℓ_1 and in ℓ_2. We have to prove that it is bounded in E, i.e., for any sequence $a = (a_k)_{k=1}^\infty \in E$, the inequality

$$\Big\| \sum_{k=1}^{\infty} (Ua)_k r_k \Big\|_X \leq D\|U\| \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X \tag{4.11}$$

holds, where $\|U\| := \max\big(\|U\|_{\ell_1\to\ell_1}, \|U\|_{\ell_2\to\ell_2}\big)$ and $D > 0$ is a constant. Due to the positive homogeneity of inequality (4.11), we can assume that $\|U\| = 1$. Introduce $x = \sum\limits_{k=1}^{\infty} a_k r_k$ and $y = \sum\limits_{k=1}^{\infty} (Ua)_k r_k$. First of all,

$$\mathcal{K}(t, Ua; \ell_1, \ell_2) \le \mathcal{K}(t, a; \ell_1, \ell_2) \ \ (t > 0).$$

Indeed, if $a = b + c$, $b \in \ell_1$ and $c \in \ell_2$, then for all $t > 0$, we have

$$\begin{aligned}\mathcal{K}(t, Ua; \ell_1, \ell_2) &\le \inf\{\|Ub\|_1 + t\|Uc\|_2 : \ a = b + c, b \in \ell_1, c \in \ell_2\} \\ &\le \inf\{\|b\|_{\ell_1} + t\|c\|_{\ell_2} : \ a = b + c, b \in \ell_1, c \in \ell_2\} = \mathcal{K}(t, a; \ell_1, \ell_2).\end{aligned}$$

Therefore, applying Corollary 2.2, we obtain the inequality $y^*(u) \le Cx^*(\beta u)$, where $C > 0$ and $\beta \in (0, 1)$ are universal constants; hence, we have

$$\|y\|_X \le C\|\sigma_{1/\beta} x^*\|_X \le \frac{C}{\beta}\|x\|_X$$

because $\|\sigma_\tau\|_{X\to X} \le \max(1, \tau)$ (see [92, Theorem 2.4.5]). Therefore, relation (4.11) is proved. □

Remark 4.1. The statement of Theorem 4.1 does not hold if the subspace G in the definition is replaced by L_q, $q < \infty$. Indeed, if we assume $(R(\ell_1), R(\ell_2))$ is a $\mathcal{K}$-subcouple of the couple (L_∞, L_q), then, due to (4.4), we have

$$\mathcal{K}(t, a; \ell_1, \ell_2) \asymp \mathcal{K}(t, Ra; L_\infty, L_q).$$

Therefore, applying the $\mathcal{K}$-method of interpolation with parameter $F = \ell_p(2^{-\theta k})$ to pairs (ℓ_1, ℓ_2) and (L_∞, L_q) $0 < \theta < 1, p = q/\theta$, we obtain (see [40, Theorem 5.2.1, p. 142]) that

$$\|Ra\|_p \asymp \|a\|_{r,p} = \left\{\sum_{k=1}^{\infty} (a_k^*)^p k^{p/r-1}\right\}^{1/p},$$

where $r = 2/(2-\theta) < 2$. On the other hand, $\|Ra\|_p \asymp \|a\|_2$ by the Khintchine theorem (Theorem 1.4). Since $\ell_2 \ne \ell_{r,p}$ if $r < 2$, it follows that the initial assumption is incorrect and the statement is proved.

Now we consider the map $X \mapsto E_X$ defined by (3.1) only on the class of interpolation spaces X with respect to the couple (L_∞, G). We want to prove that it is one-to-one in this case, i.e., the following statement holds.

Theorem 4.2. *Let X_0 and X_1 be two symmetric interpolation spaces between L_∞ and G. If*

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X_0} \asymp \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{X_1}, \tag{4.12}$$

then $X_0 = X_1$ (with the equivalence of the norms).

Denote by $\mathcal{K}(X_0 + X_1)$ the set of all $\mathcal{K}$-functionals corresponding to the Banach couple (X_0, X_1), i.e., all functions of the form $f(t) = \mathcal{K}(t, x; X_0, X_1)$ $(x \in X_0 + X_1)$. First, we prove the coincidence (accurate within equivalence) of the sets $\mathcal{K}(\ell_1 + \ell_2)$ and $\mathcal{K}(L_\infty + G)$. Further, we need two cones of quasiconcave functions defined on the semi-axis $(0, \infty)$. Define by $\mathcal{P}_0$ the set of all quasiconcave functions f such that

$$\lim_{t\to 0+} f(t) = \lim_{t\to\infty} \frac{f(t)}{t} = 0.$$

The cone $\mathcal{P}_1$ consists of all $f \in \mathcal{P}_0$ such that $f(t) = f(1)t$ $(0 < t \le 1)$. This cone appears in the study of the set of $\mathcal{K}$-functionals generated by the embedded Banach couple.

Proposition 4.1. *If (X_0, X_1) is a Banach couple such that $X_0 \subset X_1$ with the constant 1 and X_0 is everywhere dense in X_1, then $\mathcal{K}(X_0 + X_1) \subset \mathcal{P}_1$.*

Proof. Let $f(t) := \mathcal{K}(t, x; X_0, X_1)$ $(x \in X_1 = X_0 + X_1)$. Since X_0 is everywhere dense in X_1, it follows that for any $\varepsilon > 0$ we can find $x_0 \in X_0$ such that if $x_1 := x - x_0$, then $\|x_1\|_{X_1} \leq \varepsilon$. Take t_0 such that $\|x_0\|_{X_0}/t \leq \varepsilon$ holds for all $t \geq t_0$. Then $x = x_0 + x_1$ and

$$\frac{f(t)}{t} = \mathcal{K}(1/t, x; X_1, X_0) \leq \|x_1\|_{X_1} + \frac{1}{t}\|x_0\|_{X_0} \leq 2\varepsilon \quad (t \geq t_0).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lim\limits_{t\to\infty} f(t)/t = 0$. By the definition of the $\mathcal{K}$-functional, for every $x \in X_0 + X_1 = X_1$, we have

$$\mathcal{K}(t, x; X_0, X_1) = t\|x\|_{X_1} \quad (0 < t \leq 1).$$

Therefore, due to the definition of cone $\mathcal{P}_1$, the proposition is proved. □

Corollary 4.3. *The following embeddings take place*:

$$\mathcal{K}(\ell_1 + \ell_2) \subset \mathcal{P}_1 \quad \textit{and} \quad \mathcal{K}(L_\infty + G) \subset \mathcal{P}_1.$$

Now we have to prove relations that are inverse to the just obtained ones. Introduce one more definition.

Definition 4.2. Let $\mathcal{U}$ be a cone of quasiconcave functions. A Banach couple (X_0, X_1) is called *$\mathcal{U}$-full* if for any function $f \in \mathcal{U}$ there exist $x \in X_0 + X_1$ such that

$$\mathcal{K}(t, x; X_0, X_1) \asymp f(t) \ \ (t > 0).$$

In other words, the set $\mathcal{K}(X_0 + X_1)$ of all $\mathcal{K}$-functionals of a $\mathcal{U}$-full couple (X_0, X_1) contains $\mathcal{U}$ accurate within equivalence.

To prove the $\mathcal{P}_1$-completness of couples (ℓ_1, ℓ_2) and (L_∞, G), we need several auxiliary statements. The first one is the classical boundedness lemma for the Hardy–Littlewood operators

$$\mathcal{H}f(s) = \frac{1}{s}\int_0^s f(u)\,du \quad \text{and} \quad \tilde{\mathcal{H}}(a_j) = \left(\frac{1}{k}\sum_{j=1}^k a_j\right)_{k=1}^\infty$$

in L_p-spaces of functions and sequences respectively.

Lemma 4.1. *The operator $\mathcal{H}$ $(\tilde{\mathcal{H}})$ is bounded in the space $L_p = L_p(0,\infty)$ (l_p) for any $1 < p \leq \infty$.*

Proof. For $p = \infty$, the statement of the lemma is obvious both for $\mathcal{H}$ and $\tilde{\mathcal{H}}$. For $1 < p < \infty$, we apply the Minkowski inequality

$$\|\mathcal{H}f\|_p = \left\|\int_0^1 f(su)\,du\right\|_p \leq \int_0^1 \left(\int_0^\infty |f(su)|^p\,ds\right)^{1/p} du$$

$$\leq \int_0^1 \left(\frac{1}{u}\int_0^\infty |f(v)|^p\,dv\right)^{1/p} du = \frac{p}{p-1}\|f\|_p;$$

thus, the statement is proved for $\mathcal{H}$.

If $a = (a_k)_{k=1}^\infty$, then we can assume that $a_k = a_k^*$. Then, for the function $f_a = \sum\limits_{i=1}^\infty a_i\chi_{[i-1,i)}$, due to the previous inequality, we have

$$\|\tilde{\mathcal{H}}a\|_{l_p} \leq \left(\sum_{k=1}^\infty \int_{k-1}^k \left(\frac{1}{s}\int_0^s f_a(u)\,du\right)^p ds\right)^{1/p} = \|\mathcal{H}f_a\|_p \leq \frac{p}{p-1}\|f_a\|_p = \frac{p}{p-1}\|a\|_{\ell_p};$$

thus, the lemma is proved. □

In Proposition 2.1, we have found an equivalent expression (consisting of two summands) for the $\mathcal{K}$-functional $\mathcal{K}(t, f; L_1, L_2)$ corresponding to the couple $(L_1(0,\infty), L_2(0,\infty))$. Let us show that this expression can be simplified (accurate within equivalence), making it monomial.

Proposition 4.2. *With constants not depending on $f \in L_1(0,\infty) + L_2(0,\infty)$ and $t > 0$, we have*

$$\mathcal{K}(t, f; L_1, L_2) \asymp t \left(\int_{t^2}^{\infty} \left(\frac{1}{s} \int_0^s f^*(u)\, du \right)^2 ds \right)^{1/2}. \tag{4.13}$$

Moreover, for $t \geq 1$, we have

$$\mathcal{K}(t, f; L_1, L_2) \asymp t \left(\sum_{l=[t^2]}^{\infty} \frac{1}{l^2} \left(\int_0^l f^*(u)\, du \right)^2 \right)^{1/2}. \tag{4.14}$$

Proof. First of all, denote by $G(t, f)$ the right-hand side of relation (4.13), and by $F(t, f)$ the expression from (2.2), i.e.,

$$F(t, f) = \int_0^{t^2} f^*(s)\, ds + t \Big(\int_{t^2}^{\infty} f^*(s)^2\, ds \Big)^{1/2}.$$

Since $f^*(s) \leq 1/s \int\limits_0^s f^*(u)\, du$, we have

$$G(t, f) \geq t \left(\int_{t^2}^{\infty} f^*(u)^2\, du \right)^{1/2}.$$

Moreover, we have

$$G(t, f) \geq t \left(\int_{t^2}^{\infty} \left(\frac{1}{s} \int_0^{t^2} f^*(u)\, du \right)^2 ds \right)^{1/2} = \int_0^{t^2} f^*(u)\, du;$$

therefore, $2G(t, f) \geq F(t, f)$. From this inequality and (2.2), it follows that $\mathcal{K}(t, f; L_1, L_2) \leq 2G(t, f)$.

To prove the inverse relation, we estimate

$$\begin{aligned} G(t, f) &= t \left(\int_{t^2}^{\infty} \left(\frac{1}{s} \int_0^{t^2} f^*(u)\, du + \frac{1}{s} \int_{t^2}^{s} f^*(u)\, du \right)^2 ds \right)^{1/2} \\ &\leq t \left(\int_{t^2}^{\infty} \left(\frac{1}{s} \int_0^{t^2} f^*(u)\, du \right)^2 ds \right)^{1/2} + t \left(\int_{t^2}^{\infty} \left(\frac{1}{s} \int_{t^2}^{s} f^*(u)\, du \right)^2 ds \right)^{1/2} \\ &= \int_0^{t^2} f^*(u)\, du + t \left(\int_{t^2}^{\infty} \left(\frac{1}{s} \int_{t^2}^{s} f^*(u)\, du \right)^2 ds \right)^{1/2}. \end{aligned}$$

Since

$$\left(\int_{t^2}^{\infty}\left(\frac{1}{s}\int_{t^2}^{s} f^*(u)\,du\right)^2 ds\right)^{1/2} = \left(\int_{0}^{\infty}\left(\frac{1}{s+t^2}\int_{0}^{s} f^*(u+t^2)\,du\right)^2 ds\right)^{1/2}$$
$$\leq \left(\int_0^\infty \big(\mathcal{H}f^*(\cdot+t^2)(s)\big)^2\,ds\right)^{1/2},$$

where $\mathcal{H}$ is the Hardy–Littlewood operator, it follows from Lemma 4.1 that

$$G(t,f) \leq \int_0^{t^2} f^*(u)\,du + 2t\left(\int_0^\infty \big(f^*(s+t^2)\big)^2\,ds\right)^{1/2}$$
$$= \int_0^{t^2} f^*(u)\,du + 2t\left(\int_{t^2}^\infty (f^*(s))^2\,ds\right)^{1/2} \leq 2F(t,f).$$

Therefore, in view of (2.2), we have

$$G(t,f) \leq 8\mathcal{K}(t,f;L_1,L_2).$$

Thus, (4.13) is proved.

Relation (4.14) is the corollary of (4.13). Indeed, for $m \in \mathbb{N}$, we have

$$\mathcal{K}(m,f;L_1,L_2) \asymp m\left(\sum_{l=m^2}^{\infty}\int_l^{l+1}\left(\frac{1}{s}\int_0^s f^*(u)\,du\right)^2 ds\right)^{1/2}.$$

It suffices to use the concavity of the functions $\int\limits_0^s f^*(u)\,du$ and $\mathcal{K}(s,f;L_1,L_2)$. ☐

Proposition 4.3. *The Banach couple* $(L_1(0,\infty), L_2(0,\infty))$ *is* $\mathcal{P}_0$*-complete.*

Proof. We use the following general theorem from the theory of interpolation of operators (see [51, Theorem 4.5.7, p. 614]): For the $\mathcal{P}_0$-completeness of a Banach couple (X_0,X_1) it suffices that the set $\mathcal{K}(X_0+X_1)$ with some $\theta \in (0,1)$ contains a function equivalent to the function t^θ. Relation (4.13) and calculations show that if $f_0(u) = u^{-3/4}$, then $f_0 \in L_1(0,\infty)+L_2(0,\infty)$ and $\mathcal{K}(t,f_0;L_1,L_2) \asymp t^{1/2}$ $(t>0)$. ☐

Corollary 4.4. *The Banach couples* (ℓ_1,ℓ_2) *and* (L_∞, G) *are* $\mathcal{P}_1$*-complete.*

Proof. To every function $x \in L_1(0,\infty)+L_2(0,\infty)$, associate the sequence

$$a(x) = (a_k(x))_{k=1}^\infty, \quad \text{where} \quad a_k(x) = \int_{k-1}^{k} x(s)\,ds,$$

and denote by Q the averaging operator with respect to the system of intervals $\{(k-1,k]\}_{k=1}^\infty$, i.e.,

$$Qx(t) = \sum_{k=1}^\infty a_k(x)\,\chi_{(k-1,k]}(t) \ \ (t>0).$$

From Proposition 4.2 (see (4.14)) it follows that

$$\mathcal{K}(t,Qx^*;L_1,L_2) \asymp \mathcal{K}(t,x;L_1,L_2) \ \ (t \geq 1).$$

Let $f \in \mathcal{P}_1$ be chosen arbitrary. Since $\mathcal{P}_1 \subset \mathcal{P}_0$, it follows from the previous proposition that there exists a function $x \in L_1(0,\infty) + L_2(0,\infty)$ such that

$$\mathcal{K}(t, x; L_1, L_2) \asymp f(t) \ \ (t > 0).$$

At the same time, by Lemma 2.1, we have

$$\mathcal{K}(t, Qx^*; L_1, L_2) = \mathcal{K}(t, a(x^*); \ell_1, \ell_2) \ \ (t > 0).$$

Therefore, since $\|a(x)\|_{\ell_2} \le \|x\|_2$, it follows from the three last relations that there exists a sequence $a \in \ell_2$ such that

$$\mathcal{K}(t, a; \ell_1, \ell_2) \asymp f(t) \ \ (t \ge 1).$$

It holds for $0 < t \le 1$ because the following relation is valid in this case:

$$\mathcal{K}(t, a; \ell_1, \ell_2) = t\|a\|_2 = t\mathcal{K}(1, a; \ell_1, \ell_2) \asymp tf(1) = f(t).$$

Thus, the couple (ℓ_1, ℓ_2) is $\mathcal{P}_1$-complete. Since, due to Theorem 4.1, the set $\mathcal{K}(L_\infty + G)$ contains the set $\mathcal{K}(\ell_1 + \ell_2)$, accurate within equivalence, it follows that the corollary is proved. □

Now we prove the $\mathcal{K}$-monotonicity of the couple (L_∞, G). The following notion is used.

Definition 4.3. A Banach couple $\vec{X} = (X_0, X_1)$ is called a *partial retract* of the couple $\vec{Y} = (Y_0, Y_1)$ if every $x \in X_0 + X_1$ is *orbitally equivalent* to some $y \in Y_0 + Y_1$, i.e., there exist linear operators $U \in L(\vec{X}, \vec{Y})$ and $V \in L(\vec{Y}, \vec{X})$ such that $Ux = y$ and $Vy = x$.

Lemma 4.2. *If a Banach couple $\vec{X} = (X_0, X_1)$ is a partial retract of a $\mathcal{K}$-monotone Banach couple $\vec{Y} = (Y_0, Y_1)$, then the couple $\vec{X}$ is $\mathcal{K}$-monotone as well.*

Proof. Assume that $x_i \in X_0 + X_1$ $(i = 1, 2)$ and

$$\mathcal{K}(t, x_2; \vec{X}) \le \mathcal{K}(t, x_1; \vec{X}) \ \ (t > 0). \tag{4.15}$$

By the condition, there exist $y_i \in Y_0 + Y_1$ $(i = 1, 2)$ and linear operators $U_i \in L(\vec{X}, \vec{Y})$ and $V_i \in L(\vec{Y}, \vec{X})$ such that $y_i = U_i x_i$ and $x_i = V_i y_i$ $(i = 1, 2)$. Then

$$\begin{aligned}\mathcal{K}(t, x_i; \vec{X}) = \mathcal{K}(t, V_i y_i; \vec{X}) &\le \inf\{\|V_i z_0\|_{X_0} + t\|V_i z_1\|_{X_1} : \, y_i = z_0 + z_1\} \\ &\le \ \|V_i\|_{L(\vec{Y}, \vec{X})} \cdot \mathcal{K}(t, y_i; \vec{Y}) \ \ (i = 1, 2).\end{aligned}$$

Similarly, we have

$$\mathcal{K}(t, y_i; \vec{Y}) \le \|U_i\|_{L(\vec{X}, \vec{Y})} \cdot \mathcal{K}(t, x_i; \vec{X}) \ \ (i = 1, 2).$$

Therefore, from (4.15) it follows that

$$\mathcal{K}(t, y_2; \vec{Y}) \le \|U_2\|\|V_1\|\mathcal{K}(t, y_1; \vec{Y}) \ \ (t > 0).$$

Since the couple $\vec{Y}$ is $\mathcal{K}$-monotone, it follows that there exists a linear operator $W \in L(\vec{Y}, \vec{Y})$ such that $y_2 = Wy_1$. Thus, we obtain that $x_2 = Sx_1$, where $S := V_2 W U_1 \in L(\vec{X}, \vec{X})$. Therefore, the lemma is proved. □

Now we need one more definition.

Definition 4.4. Let E and F be measurable subsets of the real line of the same measure. We say that a map $\omega : E \to F$ is a *measure preserving transformation* if for any measurable $C \subset F$, its inverse image $\omega^{-1}(C)$ is measurable and the following equality holds: $m(\omega^{-1}(C)) = m(C)$.

Lemma 4.3. *Let functions $x(t)$ and $y(t)$ be measurable on $[0, 1]$ such that $|x(t)|$ and $|y(t)|$ are equimeasurable. Then there exists a linear operator T bounded in any symmetric space such that $y = Tx$.*

Proof. First, we show that for arbitrary equimeasurable functions $u(t) \geq 0$ and $v(t) \geq 0$ on $[0,1]$ and any $\delta > 0$, there exists a measure-preserving transformation $\tau : [0,1] \to [0,1]$ such that

$$\|v - u_\tau\|_\infty \leq \delta, \quad \text{where} \quad u_\tau(t) = u(\tau(t)). \tag{4.16}$$

Indeed, consider sets

$$U_k := \{t \in [0,1] : \delta k \ < \ u(t) \leq \delta(k+1)\}$$

and

$$V_k := \{t \in [0,1] : \delta k \ < \ v(t) \leq \delta(k+1)\} \ \ (k = 0, 1, \ldots).$$

Since $u(t)$ and $v(t)$ are equimeasurable, it follows that $m(U_k) = m(V_k)$; therefore, there exist a measure-preserving transformation $\tau_k : V_k \to U_k$ $(k = 0, 1, \ldots)$. Denote

$$U_{-1} := [0,1] \setminus \bigcup_{k=0}^{\infty} U_k \quad \text{and} \quad V_{-1} = [0,1] \setminus \bigcup_{k=0}^{\infty} V_k.$$

It is obvious that $u(t) = 0\,(t \in U_{-1})$, $v(t) = 0\,(t \in V_{-1})$ and $m(U_{-1}) = m(V_{-1})$. Let $\tau_{-1} : V_{-1} \to U_{-1}$ preserve measure. Define $\tau : [0,1] \to [0,1]$ with the help of "pasting" of mappings $\tau_k\,(k = -1, 0, 1, \ldots)$, i.e., $\tau(t) = \tau_k(t)$ if $t \in V_k\,(k = -1, 0, 1, \ldots)$. One can check that τ preserves measures and (4.16) holds.

Introduce

$$E_0 = \{t : |x(t)| > 1\} \quad \text{and} \quad E_n = \left\{t : \frac{1}{n+1} < |x(t)| \leq \frac{1}{n}\right\} \ \ (n \in \mathbb{N}).$$

Denote similar sets for y by F_n. Due to the equimeasurability of x and y, we have $m(E_n) = m(F_n)$. Moreover, one can easily check that $|x|\chi_{E_n}$ and $|y|\chi_{F_n}$ are equimeasurable for any $n = 0, 1, \ldots$ Therefore, as it is proved for any $\varepsilon \in (0,1)$ and $n = 0, 1 \ldots$, there exists a measure-preserving transformation $\omega_n : [0,1] \to [0,1]$ such that

$$|y|\chi_{F_n} - (|x|\chi_{E_n})(\omega_n)\|_\infty \ \leq \ \frac{\varepsilon}{n+1}. \tag{4.17}$$

It follows from the definition of E_n that $\omega_n : F_n \to E_n$.

Introduce $z_n = |y|\chi_{F_n} - (|x|\chi_{E_n})(\omega_n)$, $\alpha_n(t) = z_n(t)/(|x|\chi_{E_n})(\omega_n(t))$ if $t \in F_n$, and $\alpha_n(t) = 0$ if $t \notin F_n$. It follows from (4.17) and the definition of E_n that

$$\|\alpha_n\|_\infty \leq \varepsilon, \quad n = 0, 1, 2, \ldots \tag{4.18}$$

Moreover, we have

$$|y|\chi_{F_n} = (|x|\chi_{E_n})(\omega_n) + z_n = (1 + \alpha_n)(|x|\chi_{E_n})(\omega_n), \quad n = 0, 1, \ldots \tag{4.19}$$

Denote

$$E_{-1} := [0,1] \setminus \bigcup_{n=0}^{\infty} E_n \quad \text{and} \quad F_{-1} := [0,1] \setminus \bigcup_{n=0}^{\infty} F_n.$$

Also, let, $\alpha_{-1}(t) = 0$ for all $t \in [0,1]$. Since $m(E_{-1}) = m(F_{-1})$, it follows that there exists a map $\omega_{-1} : F_{-1} \to E_{-1}$ saving measures. Now, if $\omega : [0,1] \to [0,1]$ and $\alpha : [0,1] \to \mathbb{R}$ are generated "by pasting" from ω_n and α_n respectively, i.e., $\omega(t) = \omega_n(t)$ and $\alpha(t) = \alpha_n(t)$ if $t \in F_n$ $(n = -1, 0, 1, \ldots)$, then, due to (4.19), we have

$$|y(t)| = (1 + \alpha(t))|x(\omega(t))|, \ \ t \in [0,1]. \tag{4.20}$$

One can easily check that the map ω preserves measures of sets. Due to (4.18), we have $\|\alpha\|_\infty \leq \varepsilon$. Now, we can define the operator

$$Tu(t) = \operatorname{sign} x(\omega(t)) \operatorname{sign} y(t)(1 + \alpha(t))u(\omega(t)).$$

Then, due to the properties of ω and α, for any $\tau > 0$, we have

$$\begin{aligned} m\{t \in [0,1] : \ |Tu(t)| > \tau\} &\leq m\{t \in [0,1] : \ (1+\varepsilon)|u(\omega(t))| > \tau\} \\ &= m\{t \in [0,1] : \ (1+\varepsilon)|u(t)| > \tau\}. \end{aligned}$$

Therefore, $(Tu)^*(t) \le (1+\varepsilon)u^*(t)$ $(t \in [0,1])$ and $\|Tu\|_X \le (1+\varepsilon)\|u\|_X$ for any symmetric space X. It follows from (4.20) and the definition of the operator T that $y = Tx$. □

Remark 4.2. It follows from the Calderon–Mityagin theorem (see Sec. 0.5) that a weaker result holds: If the absolute values of functions $x(t)$ and $y(t)$ are equimeasurable on $[0,1]$, then there exists a linear operator T bounded in any symmetric interpolation space with respect to the Banach couple (L_1, L_∞) (in particular, in separable or maximal ones) such that $y = Tx$.

Remark 4.3. The proved result cannot be strengthened as follows: for an arbitrary pair of functions $x(t)$ and $y(t)$ with absolute values equimeasurable on $[0,1]$ there exists a map $\omega : [0,1] \to [0,1]$ saving measure such that $|y(t)| = |x(\omega(t))|$. An example is given in [39, Example 2.7.7]. This example shows that the statement is incorrect even if we weaken the condition on ω and consider maps that do not decrease measures, i.e., such that $m(\omega^{-1}(E)) \le m(E)$ for any measurable $E \subset [0,1]$.

Recall (see Sec. 0.2) that a dilation function $\mathcal{M}_f(t)$ of the function f defined on $(0,1)$ is given as follows:

$$\mathcal{M}_f(t) = \sup\left\{\frac{f(st)}{f(s)} : 0 < s \le \min(1, t^{-1})\right\}, \quad t > 0.$$

The numbers

$$\gamma_f = \lim_{t\to 0+} \frac{\ln \mathcal{M}_f(t)}{\ln t} \quad \text{and} \quad \delta_f = \lim_{t\to\infty} \frac{\ln \mathcal{M}_f(t)}{\ln t}$$

are called the *dilation indices* of f.

Proposition 4.4. *If $M(\varphi)$ is the Marcinkiewicz space on $[0,1]$ generated by a quasiconcave function φ with index $\gamma_\varphi > 0$, then the Banach couple $\vec{X} = (L_\infty, M(\varphi))$ is $\mathcal{K}$-monotone.*

Proof. We show that $\vec{X}$ is a partial retract of the Banach couple $\vec{Y} = (L_\infty, L_\infty(\tilde{\varphi}))$, where

$$\|x\|_{L_\infty(\tilde{\varphi})} = \sup_{0<t\le 1} \tilde{\varphi}(t)|x(t)|, \quad \tilde{\varphi}(t) = \frac{t}{\varphi(t)}.$$

Since, by the Sparr theorem (see [131]), the couple $\vec{Y}$ is $\mathcal{K}$-monotone, then, due to Lemma 4.2, the proposition will be proved.

Let $x \in M(\varphi)$ ($L_\infty + M(\varphi) = M(\varphi)$ since $L_\infty \subset M(\varphi)$). By Lemma 4.3, there exists a linear operator T_1 bounded in the couple $\vec{X}$ such that $x^* = T_1 x$. For $y \in M(\varphi)$, we define the operator

$$U_1 y(t) = \sum_{k=1}^{\infty} 2^k \int_0^{2^{-k}} y(s)ds\, \chi_{(2^{-k}, 2^{-k+1}]}(t).$$

Due to the quasiconcavity of φ and the rearrangement property (see [92, property 6°, p. 89]), we have

$$\|U_1 y\|_{L_\infty(\tilde{\varphi})} \;\le\; 2 \sup_{k=1,2,\ldots} (\varphi(2^{-k+1}))^{-1} \int_0^{2^{-k}} y^*(s)ds \;\le\; 2\|y\|_{M(\varphi)}.$$

Therefore, $U_1 \in L(\vec{X}, \vec{Y})$. Since $x^*(t)$ is non-increasing, it follows that $U_1 x^*(t) \ge x^*(t)$. Therefore, the linear operator

$$Uy(t) := \frac{x^*(t)}{U_1 x^*(t)} U_1 T_1 y(t)$$

is bounded from the couple $\vec{X}$ to the couple $\vec{Y}$ and $Ux(t) = x^*(t)$.

Let I be the identity operator, i.e., $Iy(t) := y(t)$. Since $\gamma_\varphi > 0$, it follows from [92, Theorem 2.5.3, p. 156] and the monotonicity of $\tilde{\varphi}$ and $y^*(t)$ that

$$\|Iy\|_{M(\varphi)} = \|y\|_{M(\varphi)} \;\le\; C \sup_{0<t\le 1} \tilde{\varphi}(t) y^*(t) \;\le\; C \sup_{0<t\le 1} \tilde{\varphi}(t)|y(t)| = C\|y\|_{L_\infty(\tilde{\varphi})}.$$

Therefore, $I \in L(\vec{Y}, \vec{X})$. Using Lemma 4.3 we find a linear operator T_2 bounded in the couple $\vec{X}$ such that $x = T_2 x^*$. Then the operator $V := T_2 I$ is bounded from the couple $\vec{Y}$ to couple $\vec{X}$ and $Vx^* = x$.

Therefore, an arbitrary $x \in M(\varphi)$ is orbitally equivalent to its rearrangement x^* with respect to the couples $\vec{X}$ and $\vec{Y}$ respectively and, therefore, the former is a partial retract of the latter. $\square$

Corollary 4.5. *If $\gamma_\varphi > 0$, then the couple $(L_\infty, M(\varphi)^\circ)$ is $\mathcal{K}$-monotone. In particular, the couple (L_∞, G) is $\mathcal{K}$-monotone.*

Proof. First of all, it follows from the definition of the space $M(\varphi)^\circ$ that it is interpolation with respect to the couple $(L_\infty, M(\varphi))$. Indeed, let A be a linear operator $A : L_\infty \to L_\infty$ and $A : M(\varphi) \to M(\varphi)$. If $x \in M(\varphi)^\circ$, then there exists a sequence $\{x_n\} \subset L_\infty$ such that $x_n \to x$ in the space $M(\varphi)$. Then $\{Ax_n\} \subset L_\infty$ and $Ax_n \to Ax$ in $M(\varphi)$. Therefore, $Ax \in M(\varphi)^\circ$, i.e., $A : M(\varphi)^\circ \to M(\varphi)^\circ$.

Assume that $x, y \in M(\varphi)^\circ$ and

$$\mathcal{K}(t, y; L_\infty, M(\varphi)^\circ) \;\le\; \mathcal{K}(t, x; L_\infty, M(\varphi)^\circ) \;\; (t > 0).$$

Since

$$\mathcal{K}(t, z; L_\infty, M(\varphi)^\circ) = \mathcal{K}(t, z; L_\infty, M(\varphi)) \;\; (t > 0) \tag{4.21}$$

for all $z \in M(\varphi)^\circ$, we have

$$\mathcal{K}(t, y; L_\infty, M(\varphi)) \;\le\; \mathcal{K}(t, x; L_\infty, M(\varphi)) \;\; (t > 0).$$

Therefore, due to Proposition 4.4, there exists a linear operator S bounded in the couple $(L_\infty, M(\varphi))$ and such that $y = Sx$. Then S is bounded in the couple $(L_\infty, M(\varphi)^\circ)$. The corollary is proved. $\square$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. By Lemma 3.2 the Orlicz space L_{N_2} coincides with the Marcinkiewicz space $M(\psi_2)$, where $\psi_2(u) = u \ln^{1/2}(e/u)$. Since $\gamma_\psi = 1$, it follows from Corollary 4.5 that the Banach couple (L_∞, G) is $\mathcal{K}$-monotone. Then, due to the Brudnyi–Kruglyak theorem (see [50] and Sec. 0.5), there exist parameters F_0 and F_1 of the real $\mathcal{K}$-method such that

$$X_0 = (L_\infty, G)^{\mathcal{K}}_{F_0} \quad \text{and} \quad X_1 = (L_\infty, G)^{\mathcal{K}}_{F_1}. \tag{4.22}$$

Then, by Corollary 4.1, we have

$$\left\| \sum_{k=1}^\infty a_k r_k \right\|_{X_i} \asymp \|(a_k)\|_{E_i}, \quad \text{where} \quad E_i = (\ell_1, \ell_2)^{\mathcal{K}}_{F_i} \; (i = 0, 1).$$

Therefore, due to Condition (4.12), we have

$$(\ell_1, \ell_2)^{\mathcal{K}}_{F_0} = (\ell_1, \ell_2)^{\mathcal{K}}_{F_1}. \tag{4.23}$$

The last relation means that the norms of F_0 and F_1 are equivalent on the set of $\mathcal{K}$-functionals $\mathcal{K}(\ell_1 + \ell_2)$ with respect to the couple (ℓ_1, ℓ_2). From Corollaries 4.3 and 4.4, it follows that, accurate within equivalence,

$$\mathcal{K}(\ell_1 + \ell_2) = \mathcal{K}(L_\infty + G) = \mathcal{P}_1. \tag{4.24}$$

Now let $x \in X_0$. Due to (4.22), this means that $(\mathcal{K}(2^k, x; L_\infty, G))_k \in F_0$. Applying (4.24), we find a sequence $a \in \ell_2$ such that

$$\mathcal{K}(t, a; \ell_1, \ell_2) \asymp \mathcal{K}(t, x; L_\infty, G) \;\; (t > 0).$$

Since the parameter of the $\mathcal{K}$-method is a Banach lattice, it follows that $(\mathcal{K}(2^k, a; \ell_1, \ell_2))_k \in F_0$, and due to (4.23), $(\mathcal{K}(2^k, a; \ell_1, \ell_2))_k \in F_1$. Therefore, $(\mathcal{K}(2^k, x; L_\infty, G))_k \in F_1$, i.e., $x \in X_1$. Thus, $X_0 \subset X_1$. Changing the places of X_0 and X_1 we obtain that $X_1 \subset X_0$, i.e., $X_0 = X_1$ as sets. Since every symmetric space on $[0, 1]$ is continuously embedded into $L_1[0, 1]$ (see [92, Theorem 2.4.1]), it follows from the closed graph theorem that the norms in spaces X_0 and X_1 are equivalent and Theorem 4.2 is proved. $\square$

A simple example, where $X_0 = G$ and $X_1 = L_{N_2}$, $N_2(t) = e^{t^2} - 1$, shows that the statement of Theorem 4.2 does not hold without the assumption about the interpolation X_0 and X_1 with respect to the couple (L_∞, G). Indeed, $E_G = E_{L_{N_2}} = \ell_2$, but $G \neq L_{N_2}$ (e.g., $\ln^{1/2}(e/t) \in L_{N_2} \setminus G$). Show that the conditions of Theorem 4.2 are exact and in a stronger sense.

Let X be an arbitrary symmetric space on $[0,1]$. Then, due to Corollary 4.2, the sequence space E_X with the norm defined by (3.1) is interpolation with respect to the couple (ℓ_1, ℓ_2). Therefore, $E_X = (\ell_1, \ell_2)^{\mathcal{K}}_F$ for some parameter F of the $\mathcal{K}$-method. Then, since $Y := (L_\infty, G)^{\mathcal{K}}_F$, it follows from Corollary 4.1 that

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_Y \asymp \|(a_k)\|_{E_X}.$$

Therefore, $E_Y = E_X$. Since $X \notin \mathcal{I}(L_\infty, G)$, it follows that $Y \neq X$. Therefore, for every symmetric space X not interpolational with respect to the couple (L_∞, G) there exists a unique (by theorem 4.2) symmetric space $Y \in \mathcal{I}(L_\infty, G)$ such that $E_Y = E_X$. More exact result can be proved for Marcinkiewicz spaces.

Theorem 4.3. *If ψ is a function quasiconcave on $[0,1]$ and the Marcinkiewicz space $M(\psi) \notin \mathcal{I}(L_\infty, L_{N_2})$, then there exists a Marcinkiewicz space $M(\varphi)$ such that $M(\varphi) \overset{\neq}{\subset} M(\psi)$ and $E_{M(\psi)} = E_{M(\varphi)}$.*

The following relation for the $\mathcal{K}$-functional $\mathcal{K}(t, x; L_\infty, L_{N_\alpha})$, where $N_\alpha(t) = \exp(t^\alpha) - 1$, is used below.

Lemma 4.4. *Since $\alpha \geq 1$, it follows that there exist constants independent of $x \in L_{N_\alpha}$ and $t > 0$ and such that*

$$\mathcal{K}(t, x; L_\infty, L_{N_\alpha}) \asymp t \sup_{0<u\leq \min(1, e^{1-t^\alpha})} \big(x^*(u) \ln^{-1/\alpha}(e/u)\big). \tag{4.25}$$

Proof. First of all, using (4.6) for $\mathcal{K}$-functionals in a couple of Marcinkiewicz spaces, we obtain

$$\mathcal{K}(t, x; L_\infty, L_{N_\alpha}) \asymp \|x\|_{M(\psi_t)}, \quad \text{where} \quad \psi_t(u) = u \max\{1, t^{-1} \ln^{1/\alpha}(e/u)\}. \tag{4.26}$$

Let $\mathcal{M}_{\psi_t}(s)$ be a dilation function ψ_t, i.e.,

$$\mathcal{M}_{\psi_t}(s) = \sup_{0<u\leq \min(1,1/s)} \frac{\psi_t(su)}{\psi_t(u)}.$$

We represent

$$\mathcal{M}_{\psi_t}\left(\frac{1}{2}\right) = \frac{1}{2} \sup_{0<u\leq 1} F_t(u), \quad \text{where} \quad F_t(u) = \frac{\max(1, t^{-1} \ln^{1/\alpha}(2e/u))}{\max(1, t^{-1} \ln^{1/\alpha}(e/u))}.$$

Since $\alpha \geq 1$, we have $\ln^{1/\alpha}(2e/u) \leq (1 + \ln 2) \ln^{1/\alpha}(e/u)$ $(0 < u \leq 1)$. Then for any $t > 0$, we have $F_t(u) \leq 1 + \ln 2$; hence $\mathcal{M}_{\psi_t}(1/2) \leq (1 + \ln 2)/2 < 1$. Therefore, due to [92, Theorem 2.5.3, p. 156], there exists a constant depending on α such that

$$\begin{aligned}\|x\|_{M(\psi_t)} &\asymp \sup_{0<u\leq 1} \big\{x^*(u) \min\big(1, t \cdot \ln^{-1/\alpha}(e/u)\big)\big\} \\ &= t \sup_{0<u\leq \min(1, e^{1-t^\alpha})} \big(x^*(u) \ln^{-1/\alpha}(e/u)\big).\end{aligned}$$

Using relation (4.26), we complete the proof. □

Further, we need a modification of relation (4.25).

Remark 4.4. For any $1 < a < b < \infty$, we have

$$\frac{\log_a(a/u)}{\log_a b} \le \log_b(b/u) \le \log_a(a/u) \quad (0 < u \le 1).$$

Therefore, by Lemma 3.2, we have

$$\|x\|_{L_{N_\alpha}} \asymp \sup_{0<u\le 1} \big(x^*(u) \ln^{-1/\alpha}(e/u)\big) \asymp \sup_{0<u\le 1} \big(x^*(u) \log_a^{-1/\alpha}(a/u)\big).$$

Therefore, similarly to the proof of the last lemma, we can show that

$$\mathcal{K}(t, x; L_\infty, L_{N_\alpha}) \asymp t \sup_{0<u\le \min(1, a^{1-t^\alpha})} \big(x^*(u) \log_a^{-1/\alpha}(a/u)\big) \tag{4.27}$$

for every $a > 1$ (with constants depending on a and α).

Proof of Theorem 4.3. Let $x_a = \sum_{k=1}^{\infty} a_k r_k$ and $\kappa_a(t) = \mathcal{K}(t, a; \ell_1, \ell_2)$ $(a = (a_k)_{k=1}^\infty \in \ell_2)$. By the definition of the space $E_\psi := E_{M(\psi)}$ and due to Corollary 2.2 and the boundedness of dilation operators in any symmetric space, we have

$$\|a\|_{E_\psi} = \|x_a\|_{M(\psi)} \asymp \big\|\kappa_a\big(\ln^{1/2}(e/u)\big)\big\|_{M(\psi)}. \tag{4.28}$$

Therefore,

$$\|a\|_{E_\psi} \asymp \sup_{0<t\le 1} \frac{1}{\psi(t)} \int_0^t \kappa_a\big(\ln^{1/2}(e/s)\big)\, ds$$

$$\asymp \sup_{k=1,2,..} \frac{1}{\psi(e^{-k})} \int_0^{e^{-k}} \kappa_a\big(\ln^{1/2}(e/s)\big)\, ds \asymp \sup_{k=1,2,..} \frac{1}{\psi(e^{-k})} \int_{\sqrt{k}}^{\infty} \kappa_a(u) e^{-u^2} u\, du.$$

Therefore, $E_\psi = (\ell_1, \ell_2)^{\mathcal{K}}_{F_\psi}$, where

$$\|f\|_{F_\psi} = \sup_{k=1,2,...} \frac{1}{\psi(e^{-k})} \int_{\sqrt{k}}^{\infty} |f(u)| e^{-u^2} u\, du.$$

Therefore, if $Y_\psi := (L_\infty, G)^{\mathcal{K}}_{F_\psi}$, then (see reasoning below Theorem 4.3) we have

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{M(\psi)} = \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{Y_\psi}. \tag{4.29}$$

Since the norm in Y_ψ is defined by the relation

$$\|x\|_{Y_\psi} = \sup_{k=1,2,..} \frac{1}{\psi(e^{-k})} \int_{\sqrt{k}}^{\infty} \mathcal{K}(s, x; L_\infty, G) e^{-s^2} s\, ds,$$

it follows from the concavity of the $\mathcal{K}$-functional that

$$\|x\|_{Y_\psi} \asymp \sup_{k=1,2,..} \frac{1}{\psi(e^{-k})} \sum_{i=k}^{\infty} \mathcal{K}(\sqrt{i}, x; L_\infty, G) e^{-i}. \tag{4.30}$$

On the other hand, using the boundedness of dilation operators, we obtain that

$$\|x\|_{M(\psi)} \asymp \sup_{k=1,2,...} \frac{1}{\psi(e^{-k})} \int_0^{e^{-k}} x^*(s)\, ds \asymp \sup_{k=1,2,...} \frac{1}{\psi(e^{-k})} \sum_{i=k}^{\infty} x^*(e^{-i}) e^{-i}. \tag{4.31}$$

By Lemma 4.4, we have

$$\mathcal{K}(\sqrt{i}, x; L_\infty, L_{N_2}) \asymp \sqrt{i} \sup_{0<u\le\min(1,e^{1-i})} \big(x^*(u)\ln^{-1/2}(e/u)\big)$$
$$\ge \sqrt{i}\, x^*(e^{-i})(1+i)^{-1/2} \ge \frac{1}{\sqrt{2}} x^*(e^{-i}) \quad (i = 1, 2, \ldots).$$

Therefore, comparing (4.30) and (4.31), we obtain the embedding

$$Y_\psi \ \subset \ M(\psi). \tag{4.32}$$

Therefore, if φ_ψ is a fundamental function of Y_ψ, then

$$\varphi_{M(\psi)}(h) \le C\varphi_\psi(h) \ \ (0 < h \le 1). \tag{4.33}$$

Let us find $\varphi_\psi(h)$. Using Lemma 4.4, we obtain that for all $0 < h \le 1$, we have

$$\mathcal{K}(\sqrt{i}, \chi_{(0,h)}; L_\infty, L_{N_2}) \asymp \sqrt{i}\ln^{-1/2}(e/h) \quad \text{if} \quad i \le \ln(e/h)$$

and

$$\mathcal{K}(\sqrt{i}, \chi_{(0,h)}; L_\infty, L_{N_2}) \asymp 1 \quad \text{if} \quad i > \ln(e/h).$$

Since

$$\sup_{k>\ln(e/h)} \frac{1}{\psi(e^{-k})} \sum_{i=k}^{\infty} e^{-i} \asymp \sup_{k>\ln(e/h)} \frac{e^{-k}}{\psi(e^{-k})} \asymp \frac{h}{\psi(h)} = \varphi_{M(\psi)}(h),$$

it follows from (4.33) and (4.30) that

$$\varphi_\psi(h) \asymp \sup_{1\le k\le\ln(e/h)} \frac{1}{\psi(e^{-k})} \left(\ln^{-1/2}(e/h) \sum_{i=k}^{[\ln(e/h)]} \sqrt{i}e^{-i} + \sum_{i=[\ln(e/h)]+1}^{\infty} e^{-i} \right).$$

Integrating by parts, we obtain the relation

$$\int_k^\infty \sqrt{u}e^{-u}\,du = \sqrt{k}e^{-k} + \frac{1}{2}\int_k^\infty e^{-u}\frac{du}{\sqrt{u}} \le \sqrt{k}e^{-k} + \frac{1}{2}\int_k^\infty e^{-u}\sqrt{u}\,du,$$

whence

$$\int_k^\infty \sqrt{u}e^{-u}\,du \le 2e^{-k}\sqrt{k}.$$

Therefore, if $1 \le k \le \ln(e/h)$, then

$$\sum_{i=k}^{[\ln(e/h)]} \sqrt{i}e^{-i} \asymp e^{-k}\sqrt{k}$$

and, therefore,

$$\varphi_\psi(h) \asymp \sup_{1\le k\le\ln(e/h)} \frac{e^{-k}\sqrt{k}\ln^{-1/2}(e/h) + h}{\psi(e^{-k})} \asymp \sup_{1\le k\le\ln(e/h)} \frac{e^{-k}\sqrt{k}}{\ln^{1/2}(e/h)\psi(e^{-k})}.$$

Since $\varphi_{M(\psi)}(h) = \tilde{\psi}(h) := h/\psi(h)$, it follows from (4.33) that the relation

$$\varphi_{M(\psi)}(h) \asymp \varphi_\psi(h) \ \ (0 < h \le 1)$$

is equivalent to the condition

$$\frac{\sqrt{k}}{e^k\psi(e^{-k})} \le C\frac{h\ln^{1/2}(e/h)}{\psi(h)} \ \ (1 \le k \le \ln(e/h), 0 < h \le 1).$$

Substituting $v = e^{-k}$ and $u = h$, we obtain the inequality

$$\frac{\tilde{\psi}(v)}{\tilde{\psi}(u)} \le C\frac{\ln^{1/2}(e/u)}{\ln^{1/2}(e/v)} \quad (0 < u \le v < 1). \tag{4.34}$$

Applying Theorem 4 and [3, Lemma 2] to the Banach couple (L_1, L_∞), we obtain the following interpolation theorem for Marcinkiewicz spaces on $[0,1]$. If ψ_0 and ψ_1 are two quasiconcave on $[0,1]$ functions such that ψ_1/ψ_0 decreases and $\lim\limits_{t\to 0}\psi_1(t)/\psi_0(t) = \infty$, then the Marcinkiewicz space $M(\psi)$ is interpolational with respect to the couple $(M(\psi_0), M(\psi_1))$ if and only if the following inequality holds for all $0 < s$ and $t \le 1$:

$$\frac{\psi(s)}{\psi(t)} \le C\max\left(\frac{\psi_0(s)}{\psi_0(t)}, \frac{\psi_1(s)}{\psi_1(t)}\right).$$

By Lemma 3.2, we have $L_{N_\alpha} = M(t\ln^{1/\alpha}(e/t))$ $(\alpha > 0)$. Since $L_\infty = M(t)$, it follows that $M(\psi) \in \mathcal{I}(L_\infty, L_{N_2})$ if and only if

$$\frac{\psi(s)}{\psi(t)} \le C\max\left(\frac{s\ln^{1/2}(e/s)}{t\ln^{1/2}(e/t)}, \frac{s}{t}\right) \quad (0 < s, t \le 1).$$

Note that the inequality $\psi(s)/\psi(t) \le s/t$ $(0 < t \le s \le 1)$ is a corollary of the quasiconcavity of ψ. Therefore, the previous relation is equivalent to inequality (4.34). Since, by the condition, $M(\psi) \notin \mathcal{I}(L_\infty, L_{N_2})$, it follows that inequality (4.34) does not hold, whence, due to (4.33), we have

$$\sup_{0<t\le 1}\frac{\varphi_\psi(t)}{\tilde{\psi}(t)} = \infty.$$

This and (4.32) yield

$$Y_\psi \ \subset\ M(\tilde{\varphi_\psi}) \ \overset{\neq}{\subset}\ M(\psi)$$

The last result and (4.29) lead to the relation

$$\left\|\sum_{k=1}^\infty a_k r_k\right\|_{M(\psi)} = \left\|\sum_{k=1}^\infty a_k r_k\right\|_{M(\varphi_\psi)}.$$

The theorem is proved. □

Corollary 4.1 allows us to find coordinate spaces consisting of sequences of coefficients for Rademacher sums from function symmetric space. Consider some examples.

Example 4.1. The Lorentz symmetric space $\Lambda_p(\varphi)$, where φ is an increasing concave function, $\varphi(0) = 0$, and $1 \le p < \infty$, consists of all measurable $x = x(s)$ such that the following norm is finite:

$$\|x\|_{\varphi,p} = \left\{\int_0^1 (x^*(s))^p\,d\varphi(s)\right\}^{1/p}.$$

Show that for $\varphi(s) = \ln^{1-p}(e/s)$ $(1 < p < 2)$, the Rademacher system in $\Lambda_p(\varphi)$ is equivalent to the canonical basis in ℓ_p, i.e.,

$$\left\|\sum_{k=1}^\infty a_k r_k\right\|_{\varphi,p} \asymp \|(a_k)\|_p. \tag{4.35}$$

First of all, $\ell_p = (\ell_1, \ell_2)_{\theta,p}$ where $\theta = 2(p-1)/p$ (see [40, Theorem 5.2.1]). Therefore, by Corollary 4.1 for this θ, we have

$$\left\|\sum_{k=1}^\infty a_k r_k\right\|_{(L_\infty,G)_{\theta,p}} \asymp \|(a_k)\|_p. \tag{4.36}$$

Then, by Lemma 4.4, since $x \in G$, it follows that

$$\mathcal{K}(\sqrt{k}, x; L_\infty, G) = \mathcal{K}(\sqrt{k}, x; L_\infty, L_{N_2})$$
$$\asymp \sqrt{k} \sup_{0<u\le e^{1-k}} \big(x^*(u)\ln^{-1/2}(e/u)\big) \ge \frac{1}{2} x^*(e^{-k}) \ \ (k = 1, 2, \dots).$$

Therefore, using the concavity of $\mathcal{K}$-functionals, the relation $\theta p = 2(p-1)$, and the simple inequalities

$$1 + \frac{\beta}{2} x \le (1+x)^\beta \le 1 + \beta x \ \ (0 \le x, \beta \le 1), \tag{4.37}$$

we obtain (also, see the definition of interpolation spaces in Sec. 0.5) the inequality

$$\|x\|^p_{(L_\infty,G)_{\theta,p}} \ge c_p \int_1^\infty \big(t^{-\theta}\mathcal{K}(t, x; L_\infty, G)\big)^p \frac{dt}{t} = c_p \sum_{k=1}^\infty \int_{\sqrt{k}}^{\sqrt{k+1}} \big(t^{-\theta}\mathcal{K}(t, x; L_\infty, G)\big)^p \frac{dt}{t}$$
$$\ge \frac{c_p}{2^p} \sum_{k=1}^\infty x^*(e^{-k})^p \int_{\sqrt{k}}^{\sqrt{k+1}} t^{-\theta p} \frac{dt}{t} = \frac{c_p}{2^{p+1}(p-1)} \sum_{k=1}^\infty x^*(e^{-k})^p (k^{1-p} - (k+1)^{1-p})$$
$$\ge \frac{c_p}{2^{p+2}} \sum_{k=1}^\infty x^*(e^{-k})^p \frac{1}{k(k+1)^{p-1}} \ge \frac{c_p}{2^{2p+1}} \sum_{k=1}^\infty x^*(e^{-k})^p \frac{1}{k^p}.$$

On the other hand, due to (4.37), we have

$$\|x\|^p_{\varphi,p} = \int_0^1 (x^*(s))^p \, d\varphi(s) \le \sum_{k=1}^\infty x^*(e^{-k})^p (\varphi(e^{-k+1}) - \varphi(e^{-k}))$$
$$= \sum_{k=1}^\infty x^*(e^{-k})^p (k^{1-p} - (k+1)^{1-p}) \le (p-1) \sum_{k=1}^\infty x^*(e^{-k})^p \frac{1}{k^p}.$$

The continuous embedding $(L_\infty, G)_{\theta,p} \subset \Lambda_p(\varphi)$ follows from the last two inequalities, whence, due to (4.36), we obtain the inequality

$$\Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\varphi,p} \le C \|(a_k)\|_p. \tag{4.38}$$

Prove the inverse inequality. Denote $\sum\limits_{k=1}^\infty a_k r_k$ by x_a and $\sum\limits_{k=1}^\infty a_k^* \chi_{(0,2^{-k-1})}$ by f_a. Note that

$$f_a(e^{-k-1}) \ge \sum_{i=1}^k a_i^* \ \ (k = 1, 2, \dots).$$

Therefore, due to (4.37) and Proposition 1.5, we have

$$\|x_a\|^p_{\varphi,p} \ge e^{-2p} \|\sigma_{e^2} x_a\|^p_{\varphi,p} = e^{-2p} \sum_{k=1}^\infty \int_{e^{-k}}^{e^{-k+1}} x_a^*(e^{-2}s)^p \, d\varphi(s) \ge e^{-2p} \sum_{k=1}^\infty x_a^*(e^{-k-1})^p (k^{1-p} - (k+1)^{1-p})$$
$$\ge e^{-2p} 2^{-p} (p-1) \sum_{k=1}^\infty k^{-p} f_a(e^{-k-1})^p \ge c'_p \sum_{k=1}^\infty \Big(\frac{1}{k} \sum_{i=1}^k a_i^*\Big)^p \ge c'_p \|(a_k)\|^p_p.$$

Now, from (4.38), we obtain relation (4.35).

Example 4.2. Consider the exponential Orlicz spaces L_{N_α}, where $N_\alpha(t) = \exp(t^\alpha) - 1$ $(t > 0)$. Since $0 < \alpha \le 2$, it follows that $L_{N_\alpha} \supset L_{N_2}$; therefore, by Theorem 3.1 and the reasoning after it, we have

$$\left\|\sum_{k=1}^{\infty} a_k r_k\right\|_{L_{N_\alpha}} \asymp \|(a_k)\|_2.$$

Show that the following relation holds for $\alpha > 2$:

$$\left\|\sum_{k=1}^{\infty} a_k r_k\right\|_{L_{N_\alpha}} \asymp \|(a_k)\|_{\ell_{\beta,\infty}}, \qquad \frac{1}{\beta} + \frac{1}{\alpha} = 1, \tag{4.39}$$

where $\ell_{\beta,\infty}$ is the coordinate Marcinkiewicz space with the following norm:

$$\|(a_k)\|_{\ell_{\beta,\infty}} = \sup_{k=1,2,\dots} k^{1/\beta-1} \sum_{i=1}^{k} a_i^*.$$

First of all, it is known from [40, Theorem 5.2.1] that $\ell_{\beta,\infty} = (\ell_1, \ell_2)_{\theta,\infty}$ where $\theta = 2(\beta-1)/\beta$. Therefore, due to Corollary 4.1, it suffices to prove that there exists θ such that

$$(L_\infty, G)_{\theta,\infty} = L_{N_\alpha}. \tag{4.40}$$

Since $1/\beta + 1/\alpha = 1$, it follows that $\alpha = 2/\theta$. Using the definition of the space of the real $\mathcal{K}$-method and relation (4.27) (see Remark 4.4) we have the relation

$$\|x\|_{(L_\infty,G)_{\theta,\infty}} \asymp \sup_{k=0,1,2,\dots} 2^{-k\theta} 2^k \sup_{0<u\le \min(1,2^{1-2^{2k}})} \big(x^*(u) \log_2^{-1/2}(2/u)\big).$$

Changing the order of the supremum operations and applying Lemma 3.2, we obtain that

$$\begin{aligned}\|x\|_{(L_\infty,G)_{\theta,\infty}} &= \sup_{0<u\le 1} \sup_{0<k\le 2^{-1}\log_2\log_2(2/u)} \big(2^{k(1-\theta)} x^*(u) \log_2^{-1/2}(2/u)\big)\\ &\asymp \sup_{0<u\le 1} \big(x^*(u) \log_2^{-1/2}(2/u) \log_2^{(1-\theta)/2}(2/u)\big)\\ &= \sup_{0<u\le 1} \big(x^*(u) \log_2^{-1/\alpha}(2/u)\big) \asymp \|x\|_{L_{N_\alpha}}.\end{aligned}$$

Therefore, relations (4.40) and (4.39) are proved.

Example 4.3. Let $M(\varphi)$ be the Marcinkiewicz space generated by the function $\varphi(t) = t\log_2\log_2(4/t)$ $(0 < t \le 1)$. Show that

$$\left\|\sum_{k=1}^{\infty} a_k r_k\right\|_{M(\varphi)} \asymp \|(a_k)\|_{\ell_1(\log)}, \tag{4.41}$$

where $\ell_1(\log)$ is the coordinate Marcinkiewicz space with the norm

$$\|(a_k)\|_{\ell_1(\log)} = \sup_{k=1,2,\dots} \log_2^{-1}(2k) \sum_{i=1}^{k} a_i^*.$$

Due to Corollary 4.1, relation (4.41) would be proved if we find a parameter F of the $\mathcal{K}$-method such that

$$(\ell_1, \ell_2)_F^{\mathcal{K}} = \ell_1(\log) \tag{4.42}$$

and

$$(L_\infty, G)_F^{\mathcal{K}} = M(\varphi). \tag{4.43}$$

Check that (4.42) and (4.43) hold for $F = \ell_\infty(u_k)$, where $u_k = 1/(k+1)$ if $k \ge 0$ and $u_k = 1$ if $k < 0$.

First, we prove (4.42). Without loss of generality, we assume that $a_i = a_i^*$ $(i = 1, 2, \ldots)$. Then, since $\kappa_a(t) = \mathcal{K}(t, a; \ell_1, \ell_2)$, it follows from (2.2) and Proposition 2.2 that

$$\kappa_a(2^k) \;\le\; \sum_{i=1}^{2^{2k}} a_i + 2^k \Big(\sum_{i=2^{2k}+1}^{\infty} a_i^2 \Big)^{1/2} \;\le\; 4\kappa_a(2^k). \tag{4.44}$$

Assume that $R := \|a\|_{\ell_1(\log)} < \infty$. From the definition of the norm in the space $\ell_1(\log)$, it follows that

$$\sum_{i=1}^{2^{2k}} a_i \;\le\; 2R(k+1), \tag{4.45}$$

whence, in particular, $a_{2^{2k}} \le 2^{-2k+1}(k+1)R$ $(k = 0, 1, \ldots)$. Apply the last inequality to estimate the second summand in (4.44):

$$\sum_{i=2^{2k}+1}^{\infty} a_i^2 = \sum_{j=k}^{\infty} \sum_{i=2^{2j}+1}^{2^{2(j+1)}} a_i^2 \;\le\; 3\sum_{j=k}^{\infty} 2^{2j} a_{2^{2j}}^2$$

$$\le\; 12R^2 \sum_{j=k}^{\infty} 2^{-2j}(j+1)^2 \;\le\; 192R^2 \int_k^{\infty} x^2 2^{-2x}\, dx.$$

Integrating by parts, we obtain the inequality

$$\sum_{i=2^{2k}+1}^{\infty} a_i^2 \le 24^2 R^2 (k+1)^2 2^{-2k}.$$

Therefore, the second summand in the sum at (4.44) is not larger than $24R(k+1)$. Therefore, due to (4.45), we see that

$$\|a\|_{(\ell_1,\ell_2)_F^{\mathcal{K}}} = \sup_{k=0,1,..} \frac{\kappa_a(2^k)}{k+1} \;\le\; 26\|a\|_{\ell_1(\log)}.$$

Conversely, if $2^{2j} + 1 \le k \le 2^{2(j+1)}$ $(j = 0, 1, \ldots)$, then, from (4.44), it follows that

$$\sum_{i=1}^{k} a_i \;\le\; 4\kappa_a(2^{j+1}) \;\le\; 4(j+2)\|a\|_{(\ell_1,\ell_2)_F^{\mathcal{K}}} \;\le\; 4\log_2(2k)\|a\|_{(\ell_1,\ell_2)_F^{\mathcal{K}}}.$$

Then $\|a\|_{\ell_1(\log)} \le 4\|a\|_{(\ell_1,\ell_2)_F^{\mathcal{K}}}$; thus, (4.42) is proved.

Now, we consider function spaces. Due to the definition of spaces of the real $\mathcal{K}$-method and relation (4.27), we have

$$\|x\|_{(L_\infty,G)_F^{\mathcal{K}}} \asymp \sup_{k=0,1,2,..} \frac{2^k}{k+1} \sup_{0<u\le\min(1,2^{1-2^{2k}})} \big(x^*(u)\log_2^{-1/2}(2/u)\big)$$

$$\asymp \sup_{0<u\le 1} x^*(u)\log_2^{-1/2}(2/u) \sup_{0\le k\le 2^{-1}\log_2\log_2(2/u)} \frac{2^k}{k+1}$$

$$\asymp \sup_{0<u\le 1} \frac{x^*(u)}{\log_2\log_2(4/u)} \asymp \|x\|_{M(\varphi)},$$

where the last relation follows from [92, Theorem 2.5.3] (see also the proof of Lemma 3.2). Therefore, (4.43) is proved. Then (4.41) follows from (4.42) and (4.43).

Comments and References. Methods of interpolation are very important in this chapter. Theorems 4.1 and 4.2 were proved by the author in [8, 11]. The notion of $\mathcal{K}$-subcouple ($\mathcal{K}$-closed subcouple) was introduced by Peetre in [115]. Recently, it was used to solve various problems of the theory of function spaces and operators (see [1, 78, 88, 119]). Corollary 4.2 describes spaces of sequences generated by Rademacher system in functional symmetric space. It first appeared in [20]. The proof of Theorem 4.2 has a quite interesting prehistory. In 1973, a weaker form of this statement with stronger restrictions for symmetric spaces X_0 and X_1 was proved. The equivalence of fundamental functions of these spaces was established (see [123, Theorem 3.11]). Later, in [124] (see Theorem 5), the result of Theorem 4.2 under the same restrictions for X_0 and X_1 was obtained. Note that the proof of Theorem 4.2 in [8] differs from the proof given here. However, the $\mathcal{K}$-monotonicity of corresponding Banach couples and their completeness with respect to certain function cones is very important for both proofs. Proposition 4.3, which is quite simple, is contained in [6]. Applying more complicated reasoning one can prove that the statement of Proposition 4.4 holds for any couple $\vec{X} = (L_\infty, M(\varphi))$ (see [59, Theorem 1]; we give a shorter proof based on the partial retracts technique). Theorem 4.3 belongs to the author and has never been published. The relation for the $\mathcal{K}$-functional in Lemma 4.4 first appeared in [23].

Relation (4.35) is proved in [124] (earlier, a weaker statement was obtained in [111]; sequences of independent functions were considered). Note that the space $\Lambda_p(\varphi)$ $\varphi(s) = \ln^{1-p}(e/s)$ $(1 < p < 2)$ is interpolational with respect to (L_∞, L_{N_2}) (see [9, Theorem 1]). Therefore, due to relations (4.35) and (4.36) and Theorem 4.2, the relation $\Lambda_p(\varphi) = (L_\infty, G)_{\theta,p}$, where $\theta = 2(p-1)/p$, holds. Using the extrapolation description of the space $\Lambda_p(\varphi)$ (see [22, Remark 3]), we can give the following criterion for the sequence of coefficients of the series $S = \sum\limits_{k=1}^{\infty} a_k r_k$ in terms of L_p-norms of its sum $S : (a_k) \in \ell_p$, $1 < p < 2$, if and only if

$$\sum_{k=1}^{\infty} \left(\frac{\|S\|_k}{k} \right)^p < \infty.$$

Another proof of this statement (based on results of [126]) is given in [113].

Similarly, relation (4.39) first was proved in [124]; it is a well-known illustration of the behavior of Rademacher functions in spaces "close" to L_∞ (see, e.g., [117, Sec. 9] or [95, p. 94]). We provide another proof. Similarly to the end of the proof of Theorem 4.3 (see Lemma 3.2 and [3, Theorem 5]), we can show that the space L_{N_α} is interpolation with respect to the couple (L_∞, L_{N_2}) if $\alpha > 2$. Taking into account Proposition 4.4 and relation (4.21), we obtain that it is interpolation with respect to the couple (L_∞, G) as well. Therefore, by Theorem 4.2, the L_{N_α} $(\alpha > 2)$ is a unique interpolation space with respect to the couple (L_∞, G). This space has property (4.39). Finally, relation (4.41) is proved in [8].

Chapter 5

RADEMACHER FUNCTIONS AND CONES OF STEP FUNCTIONS

According to the results of the two previous chapters, a subspace of a symmetric space generated by the Rademacher system can be characterized by the corresponding symmetric space of coefficient sequences. If a space is quite "close" to L_∞, then we can describe this subspace another way as well, as a cone of step functions. We consider this possibility in this chapter.

For arbitrary symmetric space X on $[0,1]$, we denote by $q(X)$ a set of all sequences $a = (a_k)_{k=1}^{\infty}$ such that

$$\|a\|_{q(X)} = \Big\| \sum_{k=1}^{\infty} a_k^* \chi_{(0,2^{-k+1}]} \Big\|_X < \infty.$$

First of all, due to Corollary 1.8, we immediately obtain an inequality

$$\|a\|_{q(X)} \le 4\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X, \tag{5.1}$$

which holds for any symmetric space X. The question arises: When does the inverse relation hold? To solve it, we introduce the following sublinear operator defined on the set of all measurable functions:

$$\mathcal{L}_2 x(t) = \left(\frac{1}{\varphi_1(t)}\int_0^t x(s)^2\,d\varphi_1(s)\right)^{1/2},$$

where $\varphi_1(s) = \log_2^{-1}(2/s)$ $(0 < s \le 1)$. As in previous chapters, if $\alpha > 0$, then $N_\alpha(t) = \exp(t^\alpha) - 1$ $(t > 0)$ and L_{N_α} is the corresponding Orlicz space. We formulate the main result of this chapter.

Theorem 5.1. *Let X be a symmetric interpolation space on $[0,1]$ with respect to the couple (L_∞, L_{N_1}). The inequality*

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X \le C\|a\|_{q(X)}, \tag{5.2}$$

holds for some $C > 0$ depending only on X if and only if the operator $\mathcal{L}_2$ is bounded in X.

Sufficiency. Obviously we can assume that $a_k = a_k^*$ $(k = 1, 2, \dots)$. For arbitrary such sequence, we assume that

$$f_a(s) = \sum_{k=1}^{\infty} a_k \chi_{(0,2^{-k+1}]}(s) \quad \text{and} \quad x_a(s) = \sum_{k=1}^{\infty} a_k r_k(s). \tag{5.3}$$

Prove that there exists a universal constant (does not depending on X) such that

$$\|\mathcal{L}_2 f_a\|_X \asymp \|x_a\|_X. \tag{5.4}$$

Since

$$\mathcal{L}_2 x^*(t) = \left(\frac{1}{\varphi_1(t)}\int_0^{\varphi_1(t)} \big(x^*(\varphi_1^{-1}(s))\big)^2\,ds\right)^{1/2}$$

it follows that the function $\mathcal{L}_2 f_a(s)$ decreases. Therefore, if

$$q_a(s) = \sum_{k=1}^{\infty} (\mathcal{L}_2 f_a)(2^{-k})\chi_{(2^{-k},2^{-k+1}]}(s) \quad \text{and} \quad y_a(s) = \sum_{k=1}^{\infty} x_a^*(2^{-k})\chi_{(2^{-k},2^{-k+1}]}(s),$$

then

$$\sigma_{1/2}(q_a)(s) \le (\mathcal{L}_2 f_a)(s) \le q_a(s) \quad \text{and} \quad \sigma_{1/2}(y_a)(s) \le x_a^*(s) \le y_a(s) \ \ (0 < s \le 1), \tag{5.5}$$

where σ_τ is the dilatation operator. Since

$$f_a(s) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} a_j\right)\chi_{(2^{-k},2^{-k+1}]}(s), \tag{5.6}$$

we have

$$\begin{aligned}\mathcal{L}_2 f_a(2^{-k}) &= \left(\frac{1}{\varphi_1(2^{-k})}\sum_{i=k+1}^{\infty}\left(\sum_{j=1}^{i} a_j\right)^2\big(\varphi_1(2^{-i+1}) - \varphi_1(2^{-i})\big)\right)^{1/2} \\ &\asymp \sqrt{k}\left(\sum_{i=k}^{\infty}\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^2\right)^{1/2} \qquad (k = 1, 2, \dots).\end{aligned} \tag{5.7}$$

On the other hand, due to Corollary 2.2, the boundedness of dilatation operators, Lemma 2.1, and Corollary 4.2, we have

$$
\begin{aligned}
x_a^*(2^{-k}) &\asymp \mathcal{K}(\sqrt{k}, a; \ell_1, \ell_2) = \mathcal{K}\Big(\sqrt{k}, \sum_{i=1}^{\infty} a_i \chi_{(i-1,i)}; L_1(0,\infty), L_2(0,\infty)\Big) \\
&\asymp \sqrt{k}\left(\sum_{i=k}^{\infty}\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^2\right)^{1/2} \qquad (k = 1, 2, \dots).
\end{aligned} \tag{5.8}
$$

Finally, (5.4) follows from relations (5.5)–(5.8) and from the inequality $\|\sigma_2\|_{X\to X} \le 2$ (note that the constants of all equivalences used are universal).

If an operator $\mathcal{L}_2$ is bounded in a symmetric space X, then (5.2) follows from (5.4) and the introduced notation. Therefore, the sufficiency is proved. ☐

The proof of the necessity is more complicated. First of all, note that, due to (5.4), from Condition (5.2) follows the boundedness in X of the restriction of the operator $\mathcal{L}_2$ to the cone W_X consisting of functions $v = v(s) \in X$ of the form $v(s) = \sum\limits_{k=1}^{\infty} b_k \chi_{(0,2^{-k+1}]}(s)$, $b_1 \ge b_2 \ge \dots \ge 0$. Show that, under certain conditions for symmetric space, the operator is bounded on the whole space.

Denote a subadditive operator

$$
\mathcal{L}x(t) = \frac{1}{\varphi_1(t)} \int_0^t x^*(s)\, d\varphi_1(s)
$$

where $\varphi_1(s) = \log_2^{-1}(2/s)$.

Proposition 5.1. *Assume that $\mathcal{L}$ is bounded in a symmetric space X. Then an operator A is bounded in X if A has the following properties*:

(a) *if $0 \le x(t) \le y(t)$ $(0 < t \le 1)$, then $0 \le Ax(t) \le Ay(t)$ $(0 < t \le 1)$*;
(b) $|Ax(t)| \le Ax^*(t)$ $(0 < t \le 1)$;
(c) *there exists $C > 0$ such that $\|Av\|_X \le C\|v\|_X$ $(v \in W_X)$.*

Proof. For any $x \in X$, define a step function

$$
u_x(t) = \sum_{k=1}^{\infty} (\mathcal{L}x)(2^{-k}) \chi_{(2^{-k}, 2^{-k+1}]}(t) \quad (0 < t \le 1).
$$

Since $\varphi_1(t)$ increases and

$$
\mathcal{L}x(t) = \frac{1}{\varphi_1(t)} \int_0^{\varphi_1(t)} x^*\big(\varphi_1^{-1}(s)\big)\, ds, \tag{5.9}
$$

it follows that $\mathcal{L}x(t)$ decreases on $[0, 1]$. Therefore, we have

$$
x^*(t) \le \mathcal{L}x(t) \le u_x(t) \quad (0 < t \le 1) \quad \text{and} \quad u_x(t) \le \mathcal{L}x(t/2) \quad (0 < t \le 1). \tag{5.10}
$$

Then, by condition, we have

$$
\|Ax\|_X \le \|Ax^*\|_X \le \|A\mathcal{L}x\|_X \le \|Au_x\|_X. \tag{5.11}
$$

Relation (5.9) shows that the sequence $\big\{\mathcal{L}x(2^{-k})\big\}_{k=1}^{\infty}$ increases and the sequence $\big\{\frac{1}{k}\mathcal{L}x(2^{-k})\big\}_{k=1}^{\infty}$ decreases. Therefore, due to [92, Theorem 2.1.1], there exists an increasing sequence $(c_k)_{k=1}^{\infty}$ such that the sequence $b_k := c_k - c_{k-1}$ $(c_0 = 0)$ decreases and

$$
\mathcal{L}x(2^{-k}) \le c_k \le 2\mathcal{L}x(2^{-k}), \quad k = 1, 2, \dots \tag{5.12}
$$

Consider a function

$$z(t) = \sum_{k=1}^{\infty} c_k \chi_{(2^{-k}, 2^{-k+1}]}(t) = \sum_{k=1}^{\infty} b_k \chi_{(0, 2^{-k+1}]}(t).$$

By condition, $\mathcal{L}$ is bounded in X. Therefore, due to (5.12), the second inequality from (5.10) and the inequality $\|\sigma_2\|_{X \to X} \leq 2$, we obtain the inequality

$$\|z\|_X \leq 2\|u_x\|_X \leq 2\|\sigma_2 \mathcal{L} x\|_X \leq 4\|\mathcal{L}\|\|x\|_X.$$

Therefore, since the sequence (b_k) decreases, it follows that $z \in W_X$. Finally, from (5.10)–(5.12), the previous inequality, and the conditions of the proposition, it follows that

$$\|Ax\|_X \leq \|Ax^*\|_X \leq \|Az\|_X \leq C\|z\|_X \leq 4C\|\mathcal{L}\|\|x\|_X.$$

□

Therefore, the proof of Theorem 5.1 would be finished if we show the boundedness of the operator $\mathcal{L}$ in symmetric space $X \in \mathcal{I}(L_\infty, L_{N_1})$ under Condition (5.2). The scheme of the the proof is as follows.

First, define some symmetric space Y (depending on X) such that $\mathcal{L}$ is bounded in Y. Then, using condition (5.2) and interpolation properties of Banach couples, we obtain that $q(Y) = q(X)$. Finally, prove that the mapping $X \mapsto q(X)$ is one-to-one on the set $\mathcal{I}(L_\infty, L_{N_1})$. Therefore, if $Y \in \mathcal{I}(L_\infty, L_{N_1})$ (by condition, X has this property), then we obtain that $X = Y$; the proof is finished.

Let E_X be the space of sequences $a = (a_k)_{k=1}^{\infty}$ with the norm

$$\|a\|_{E_X} = \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X.$$

Since $E_X \in \mathcal{I}(\ell_1, \ell_2)$ by Corollary 4.2, it follows that

$$E_X = (\ell_1, \ell_2)_F^{\mathcal{K}} \tag{5.13}$$

for some parameter of $\mathcal{K}$-method F. Consider a symmetric space

$$Y = (L_\infty, \Lambda_2)_F^{\mathcal{K}}, \tag{5.14}$$

where Λ_2 denotes the power Lorentz space $\Lambda_2(\varphi_1)$ with the norm

$$\|x\|_{\Lambda_2} = \left(\int_0^1 \big(x^*(t)\big)^2 \, d\varphi_1(t) \right)^{1/2},$$

where $\varphi_1(t) = \log_2^{-1}(2/t)$.

Now we give another example of using the partial retracts technique (see Lemma 4.2 and Proposition 4.4). We prove that the interpolation in the Banach couple $(L_\infty, \Lambda_p(\varphi))$ is described by the $\mathcal{K}$-method. We need this to prove the boundedness of $\mathcal{L}_2$ in Y.

Proposition 5.2. *Let φ be an arbitrary continuous increasing concave function on $[0,1]$, $\varphi(0) = 0$, and $1 \leq p < \infty$. Then the Banach couple $\vec{X} = (L_\infty, \Lambda_p(\varphi))$ is $\mathcal{K}$-monotone.*

Proof. Due to the Sparr theorem (see [131]) and Lemma 4.2, it suffices to show that $\vec{X}$ is a partial retract of the Banach couple $\vec{Y} = (L_\infty, L_p(\varphi'))$, where we denote by $L_p(\varphi')$ the weighted L_p-space with the norm

$$\|x\|_{L_p(\varphi')} = \left(\int_0^1 |x(t)|^p \varphi'(t) \, dt \right)^{1/p}.$$

Let $x \in \Lambda_p(\varphi)$ (since $L_\infty \subset \Lambda_p(\varphi)$, it follows that $L_\infty + \Lambda_p(\varphi) = \Lambda_p(\varphi)$). By Lemma 4.3, there exists a linear operator T_1 bounded in $\vec{X}$ such that $x^* = T_1 x$. Moreover, one can easily check that the identity operator $Iy = y$ from the couple $\vec{X}$ to the couple $\vec{Y}$ is bounded. Indeed, since $\varphi'(t)$ monotonely decreases, due to rearrangement properties (see [92, property 13°, p. 94]), we have

$$\|Iy\|_{L_p(\varphi')} = \left(\int_0^1 |y(t)|^p \varphi'(t)\,dt \right)^{1/p} \le \left(\int_0^1 y^*(t)^p\,d\varphi(t) \right)^{1/p} = \|y\|_{\Lambda_p(\varphi)}.$$

Therefore, the operator $U := IT_1$ is bounded from $\vec{X}$ to $\vec{Y}$ and $Ux = x^*$.

Without loss of generality, we can assume that $\varphi(1) = 1$. Choose numbers $1 = b_0 > b_1 > b_2 > \cdots > 0$ such that $\varphi(b_n) = 2^{-n}$ $(n = 0, 1, \dots)$ and, for all $y \in L_p(\varphi') = L_\infty + L_p(\varphi')$, denote an operator

$$V_1 y(s) = \sum_{n=1}^{\infty} 2^{n+1} \Big(\int_{b_{n+1}}^{b_n} y(t)\varphi'(t)\,dt \Big) \chi_{[b_n, b_{n-1})}(s).$$

It is bounded in L_∞. Moreover, due to the Hölder inequality, we have

$$\begin{aligned} |V_1 y(s)|^p &\le \sum_{n=1}^{\infty} 2^{(n+1)p} \Big(\int_{b_{n+1}}^{b_n} |y(t)|\varphi'(t)\,dt \Big)^p \chi_{[0, b_{n-1})}(s) \\ &\le \sum_{n=1}^{\infty} 2^{n+1} \Big(\int_{b_{n+1}}^{b_n} |y(t)|^p \varphi'(t)\,dt \Big) \chi_{[0, b_{n-1})}(s). \end{aligned}$$

Since the last sum does not increase on $s \in [0, 1]$, it follows that

$$\|V_1 y\|^p_{\Lambda_p(\varphi)} \le 4 \sum_{n=1}^{\infty} \int_{b_{n+1}}^{b_n} |y(t)|^p \varphi'(t)\,dt = 4\|y\|^p_{L_p(\varphi')},$$

i.e., V_1 from $L_p(\varphi')$ to $\Lambda_p(\varphi)$ is bounded.

Since $x^*(t)$ does not increase, it follows that $V_1 x^*(s) \ge x^*(s)$ for all $0 < s \le 1$. Therefore, a linear operator

$$V_2 y(s) = \frac{x^*(s)}{V_1 x^*(s)} V_1 y(s) \quad (0 < s \le 1)$$

from the couple $\vec{Y}$ to the couple $\vec{X}$ is bounded and $V_2 x^*(t) = x^*(t)$. Applying Lemma 4.3, we find a linear operator T_2 bounded in the couple $\vec{X}$ such that $x = T_2 x^*$. Then an operator $V := T_2 V_2$ is bounded from the couple $\vec{Y}$ to the couple $\vec{X}$ and $Vx^* = x$.

Therefore, an arbitrary $x \in \Lambda_p(\varphi)$ is orbitally equivalent to its rearrangement x^* with respect to the couples $\vec{X}$ and $\vec{Y}$ respectively and, therefore, $\vec{X}$ is a partial retract of $\vec{Y}$. $\square$

Lemma 5.1. *If the symmetric space $Z \in \mathcal{I}(L_\infty, \Lambda_2)$, then the operator $\mathcal{L}$ is bounded in Z.*

Proof. Due to the previous proposition, (L_∞, Λ_2) is $\mathcal{K}$-monotone; therefore, by the Brudnyi–Kruglyak theorem, $Z = (L_\infty, \Lambda_2)^{\mathcal{K}}_H$ for some parameter H of the real method. On the other hand, one can easily check that the $\mathcal{K}$-method "interpolates" not only linear but sublinear operators as well [50, Sec. 11]. Therefore, the boundedness of $\mathcal{L}$ in the space Z follows from its boundedness in L_∞ and Λ_2. The first statement is obvious and we have to prove only the second one.

Since the function $\mathcal{L}x(t)$ decreases, the following inequality holds:

$$\|\mathcal{L}x\|_{\Lambda_2} \le \Big\| \sum_{k=1}^{\infty} (\mathcal{L}x)(2^{-k}) \chi_{(2^{-k}, 2^{-k+1}]} \Big\|_{\Lambda_2} \le \left(\sum_{k=1}^{\infty} \big((\mathcal{L}x)(2^{-k}) \big)^2 \frac{1}{k^2} \right)^{1/2}.$$

Due to the relation

$$\|(a_k)\|_2 = \sup\left\{\sum_{k=1}^{\infty} a_k b_k : \ \|(b_k)\|_2 \le 1\right\},$$

there exists a sequence (d_k) such that $d_k \ge 0$, $\|(d_k)\|_2 \le 1$, and

$$\left(\sum_{k=1}^{\infty}\big((\mathcal{L}x)(2^{-k}\big)^2\frac{1}{k^2}\right)^{1/2} \le 2\sum_{k=1}^{\infty}\frac{d_k}{k}(\mathcal{L}x)(2^{-k}).$$

Therefore, using the definition of the operator $\mathcal{L}$, we obtain

$$\|\mathcal{L}x\|_{\Lambda_2} \le 4\sum_{k=1}^{\infty} d_k \int\limits_0^{2^{-k}} x^*(s)\,d\varphi_1(s) = 4\sum_{k=1}^{\infty}\int\limits_{2^{-k-1}}^{2^{-k}}\left(\sum_{j=1}^{k} d_j\right) x^*(s)\,d\varphi_1(s)$$

$$\le 4\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{j=1}^{k} d_j\right) x^*(2^{-k-1})\frac{1}{k}.$$

Then, due to Lemma 4.1 and the boundedness of dilatation operators in symmetric spaces, we have

$$\|\mathcal{L}x\|_{\Lambda_2} \le 4\|\tilde{\mathcal{H}}\|_{\ell_2\to\ell_2}\left(\sum_{k=1}^{\infty}\big(x^*(2^{-k-1})\big)^2\frac{1}{k^2}\right)^{1/2} \le 8\|\tilde{\mathcal{H}}\|\Big\|\sum_{k=1}^{\infty} x^*(2^{-k-1})\chi_{(2^{-k},2^{-k+1}]}\Big\|_{\Lambda_2}$$

$$= 8\|\tilde{\mathcal{H}}\|\Big\|\sum_{k=1}^{\infty}(\sigma_4 x^*)(2^{-k+1})\chi_{(2^{-k},2^{-k+1}]}\Big\|_{\Lambda_2} \le 8\|\tilde{\mathcal{H}}\|\|\sigma_4 x^*\|_{\Lambda_2} \le 32\|\tilde{\mathcal{H}}\|\|x\|_{\Lambda_2},$$

where $\tilde{\mathcal{H}}(a_j) = \left(\frac{1}{k}\sum\limits_{j=1}^{k} a_j\right)_{k=1}^{\infty}$ is the Hardy operator. □

Since $Y \in \mathcal{I}(L_\infty, \Lambda_2)$ (see (5.14)), it follows from Lemma 5.1 that the operator $\mathcal{L}$ is bounded in Y. Now we have to prove that

$$q(Y) = q(X). \tag{5.15}$$

First of all, due to relations (5.1) and (5.2) and the definition of E_X, we have

$$q(X) = E_X. \tag{5.16}$$

Therefore, to prove (5.15), it suffices to show that

$$q(Y) = E_X. \tag{5.17}$$

Let us continue studying the Banach couple (L_∞, Λ_2). In the following lemma, functions defined for $t \in [0,1]$ are extended by zero for $t > 1$.

Lemma 5.2. *There exist constants not depending on $x \in \Lambda_2$ and $t > 0$ such that*

$$\mathcal{K}(t, x; L_\infty, \Lambda_2) \asymp t\left(\int\limits_0^{2^{1-t^2}} x^*(s)^2\,d\varphi_1(s)\right)^{1/2}.$$

Proof. For a fixed $t > 0$, we define a function $x_1(s)$:

$$x_1(s) = x(s) - x^*(2^{1-t^2})x(s)/|x(s)|, \quad \text{if} \quad |x(s)| \ge x^*(2^{1-t^2})$$

and

$$x_1(s) = 0, \quad \text{if} \quad |x(s)| < x^*(2^{1-t^2}).$$

Moreover, let $x_0 := x - x_1$. Then, since $E = \{s \in [0,1] : x_1(s) \neq 0\}$, by the definition of a rearrangement, we have $x_1^*(s) = x^*(s) - x^*(2^{1-t^2})$ $(0 < s < m(E))$ and $m(E) \leq 2^{1-t^2}$. Therefore, the following inequality holds:

$$\mathcal{K}(t, x; L_\infty, \Lambda_2) \leq t\|x_1\|_{\Lambda_2} + \|x_0\|_\infty \leq t \left(\int_0^{2^{1-t^2}} (x^*(s) - x^*(2^{1-t^2}))^2 \, d\varphi_1(s) \right)^{1/2} + x^*(2^{1-t^2})$$

$$= t \left(\int_0^{2^{1-t^2}} (x^*(s) - x^*(2^{1-t^2}))^2 \, d\varphi_1(s) \right)^{1/2} + t \left(\int_0^{2^{1-t^2}} (x^*(2^{1-t^2}))^2 d\varphi_1(s) \right)^{1/2}$$

$$\leq 2t \left(\int_0^{2^{1-t^2}} (x^*(s))^2 d\varphi_1(s) \right)^{1/2}.$$

To prove the inverse inequality, we consider an arbitrary representation $x = x_0 + x_1$, where $x_0 \in L_\infty$, $x_1 \in \Lambda_2$. As in the proof of Proposition 2.1 (see (2.3)), we have $x^*(s) \leq x_0^*(s/2) + x_1^*(s/2)$, $s \in [0,1]$. Therefore, we have

$$\left(\int_0^{2^{1-t^2}} (x^*(s))^2 \, d\varphi_1(s) \right)^{1/2} \leq \left(\int_0^{2^{1-t^2}} (x_0^*(s/2))^2 \, d\varphi_1(s) \right)^{1/2}$$

$$+ \left(\int_0^{2^{1-t^2}} (x_1^*(s/2))^2 \, d\varphi_1(s) \right)^{1/2} \leq \frac{\|x_0\|_\infty}{t} + \sqrt{2}\|x_1\|_{\Lambda_2},$$

whence, by the definition of the $\mathcal{K}$-functional, the following inequality holds:

$$t \left(\int_0^{2^{1-t^2}} (x^*(s))^2 \, d\varphi_1(s) \right)^{1/2} \leq \sqrt{2}\mathcal{K}(t, x; L_\infty, \Lambda_2).$$

□

To complete the proof, we need an equivalent expression for the $\mathcal{K}$-functional of the couple $(L_\infty, \Lambda(\varphi_1))$, where $\Lambda(\varphi_1)$ is the Lorentz space generated by a function φ_1. We omit the proof of this result because it is similar to the one just mentioned.

Lemma 5.3. *There exist constants not depending on $x \in \Lambda(\varphi_1)$ and $t > 0$ such that*

$$\mathcal{K}(t, x; L_\infty, \Lambda(\varphi_1)) \asymp t \int_0^{2^{1-t}} x^*(s) \, d\varphi_1(s).$$

To prove (5.17), we need one more statement.

Proposition 5.3. *There exist constants not depending on $a = (a_k)_{k=1}^\infty \in \ell_2$ and $t > 0$ such that*

$$\mathcal{K}(t, f_a; L_\infty, \Lambda_2) \asymp \kappa_a(t), \tag{5.18}$$

where the function f_a is defined by the first relation of (5.3) and $\kappa_a(t) = \mathcal{K}(t, a; \ell_1, \ell_2)$.

Proof. Without loss of generality, we can assume that $a_k = a_k^*$ $(k = 1, 2, \dots)$. First of all, taking into account (5.6), by Lemma 5.2, we obtain that

$$\mathcal{K}(m, f_a; L_\infty, \Lambda_2) \asymp m \left(\sum_{k=m^2}^{\infty} \left(\sum_{i=1}^{k} a_i \right)^2 \int_{2^{-k}}^{2^{-k+1}} d\varphi_1(s) \right)^{1/2}$$

$$\asymp m \left(\sum_{k=m^2}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} a_i \right)^2 \right)^{1/2} \quad (m = 1, 2, \dots).$$

On the other hand, as in the proof of the sufficiency, due to Lemma 2.1 and Proposition 4.2, we have

$$\kappa_a(t) = \mathcal{K}\Big(t, \sum_{i=1}^{\infty} a_i \chi_{(i-1,i)}; L_1(0,\infty), L_2(0,\infty)\Big) \asymp t \left(\sum_{i=[t^2]}^{\infty} \left(\frac{1}{i} \sum_{j=1}^{i} a_j \right)^2 \right)^{1/2} \quad (t \geq 1).$$

The comparison of the last relations and concavity of $\mathcal{K}$-functional lead to (5.18) for $t \geq 1$.

If $0 < t < 1$, then we need the relation

$$q(\Lambda_2) = \ell_2 \quad \text{(with equivalence of norms).} \tag{5.19}$$

Indeed, due to (5.6) and the definition of the function $\varphi_1(s)$, we have

$$\|f_a\|_{\Lambda_2} = \left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} a_j \right)^2 \int_{2^{-k}}^{2^{-k+1}} d\varphi_1(s) \right)^{1/2} \asymp \left(\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{j=1}^{k} a_j \right)^2 \right)^{1/2}.$$

Since $a_k \leq 1/k \sum_{j=1}^{k} a_j$, we have the inequality $\|a\|_2 \leq C_1 \|f_a\|_{\Lambda_2}$. On the other hand, by Lemma 4.1, we have the inequality $\|f_a\|_{\Lambda_2} \leq C_2 \|\tilde{\mathcal{H}}\|_{\ell_2 \to \ell_2} \|a\|_2$, where $\tilde{\mathcal{H}} a = \left(\frac{1}{k} \sum_{j=1}^{k} a_j \right)_{k=1}^{\infty}$ is the Hardy operator.

Finally, applying (5.19) and the inequalities $\|a\|_2 \leq \|a\|_1$ and $\|x\|_{\Lambda_2} \leq \|x\|_\infty$, we obtain that

$$\mathcal{K}(t, f_a; L_\infty, \Lambda_2) = t \|f_a\|_{\Lambda_2} \asymp t \|a\|_2 = \kappa_a(t),$$

for $0 < t < 1$ and, therefore, the proposition is proved. □

Corollary 5.1. *There exist constants not depending on $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $t > 0$ such that*

$$\mathcal{K}(t, x_a; L_\infty, G) \asymp \mathcal{K}(t, f_a; L_\infty, \Lambda_2),$$

where f_a and x_a are defined by relations (5.3) and G is a separable part of the Orlicz space L_{N_2}, $N_2(u) = e^{u^2} - 1$. Moreover, if $X = (L_\infty, G)_F^{\mathcal{K}}$, where F is a Banach ideal space of two-sided sequences, then

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X \asymp \|(a_k)\|_{q(Y)},$$

where $Y = (L_\infty, \Lambda_2)_F^{\mathcal{K}}$.

Proof. The first relation follows from the proposition just proved and Theorem 4.1. The second one follows from the definition of spaces of the $\mathcal{K}$-method of interpolation. □

The last statement allows us to prove relation (5.17). Indeed, let

$$X_0 = (L_\infty, G)_F^{\mathcal{K}},$$

where the Banach frame F is defined in (5.13). Due to corollaries 5.1 and 4.1, we have

$$\|a\|_{q(Y)} \asymp \left\|\sum_{k=1}^{\infty} a_k r_k\right\|_{X_0} \asymp \left\|\sum_{k=1}^{\infty} a_k r_k\right\|_{X} = \|a\|_{E_X}.$$

Thus, (5.17) and, therefore, (5.16) are proved.

We need to prove that, from (5.16), it follows that the spaces X and Y coincide with each other. To do that, we have to represent $q(Z)$ as an interpolation space with respect to the couple (ℓ_1, ℓ_∞), provided that $Z \in \mathcal{I}(L_\infty, L_{N_1})$.

Proposition 5.4. *There exist constants that do not depend on $a = (a_k)_{k=1}^{\infty} \in \ell_\infty$ and $t > 0$ such that*

$$\mathcal{K}(t, f_a; L_\infty, L_{N_1}) \asymp \mathcal{K}(t, a, \ell_1, \ell_\infty).$$

Proof. Without loss of generality, we can assume that $a_k = a_k^*$ $(k = 1, 2, \dots)$. First, we check that

$$\|f_a\|_{N_1} \asymp \|a\|_\infty. \tag{5.20}$$

Since

$$f_a(s) \le \sum_{k=1}^{\infty} k\|a\|_\infty \chi_{(2^{-k}, 2^{-k+1}]}(s), \quad 0 < s \le 1,$$

it follows that

$$\int_0^1 \left\{\exp\left(\frac{f_a(s)}{\|a\|_\infty} \ln(4/3)\right) - 1\right\} ds \le \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-k+1}} \{\exp(k \ln(4/3)) - 1\}\, ds = 1.$$

Therefore, $\|f_a\|_{N_1} \le \dfrac{1}{\ln(4/3)}\|a\|_\infty$. On the other hand, if for some $\lambda > 0$, the inequality

$$\int_0^1 \left\{\exp\left(\frac{f_a(s)}{\lambda}\right) - 1\right\} ds \le 1$$

holds, then, due to the inequality $f_a(s) \ge \|a\|_\infty$ $(0 < s \le 1)$, we obtain that $\exp(\|a\|_\infty/\lambda) \le 2$, i.e., $\lambda \ge \dfrac{1}{\ln 2}\|a\|_\infty$. Hence, by definition of the norm in Orlicz spaces, we have

$$\|f_a\|_{N_1} \ge \frac{1}{\ln 2}\|a\|_\infty;$$

thus, we obtain (5.20).

Then, due to (5.6) and Lemma 4.4 (see also Remark 4.4), we have

$$\mathcal{K}(m, f_a; L_\infty, L_{N_1}) \asymp m \sup_{i \ge m} \left(\frac{1}{i}\sum_{j=1}^{i} a_j\right), \quad m = 1, 2, \dots$$

Since the sequence $\left(1/i \sum\limits_{j=1}^{i} a_j\right)_{i=1}^{\infty}$ decreases, it follows that

$$\mathcal{K}(m, f_a; L_\infty, L_{N_1}) \asymp \sum_{j=1}^{m} a_j, \quad m = 1, 2, \dots$$

Therefore, due to the easily checked relation

$$\mathcal{K}(t, a; \ell_1, \ell_\infty) = \sum_{i=1}^{[t]} a_i + (t - [t]) a_{[t]+1} \quad (t > 0), \tag{5.21}$$

(see also [40, Theorem 5.2.1]), we obtain that

$$\mathcal{K}(m, f_a; L_\infty, L_{N_1}) \asymp \mathcal{K}(m, a, \ell_1, \ell_\infty), \quad m = 1, 2, \ldots$$

Using the concavity of the $\mathcal{K}$-functional, one can extend the last relation to the case where $t \geq 1$. If $0 < t \leq 1$, then from the inequalities $\|a\|_\infty \leq \|a\|_1$ and $\|x\|_{N_1} \leq \|x\|_\infty$ and from (5.20), it follows that

$$\mathcal{K}(t, f_a; L_\infty, L_{N_1}) = t\|f_a\|_{N_1} \asymp t\|a\|_\infty = \mathcal{K}(t, a, \ell_1, \ell_\infty).$$

□

From the statement just proved and the definitions of spaces of the real $\mathcal{K}$-method, we obtain the following assertion:

Corollary 5.2. *Assume that $Z = (L_\infty, L_{N_1})_F^{\mathcal{K}}$ for some parameter F of the $\mathcal{K}$-method. Then*

$$q(Z) = (\ell_1, \ell_\infty)_F^{\mathcal{K}} \ \textit{(with equivalence of norms)}.$$

The following statement shows that the map $Z \mapsto q(Z)$ restricted to the set of all symmetric interpolation spaces with respect to the couple (L_∞, L_{N_1}) is one-to-one (it is interesting to compare this result with Theorem 4.2).

Proposition 5.5. *Let spaces X_0 and X_1 be interpolation with respect to a Banach couple (L_∞, L_{N_1}) and $q(X_0) = q(X_1)$. Then $X_0 = X_1$ (with equivalence of norms).*

Proof. Denote by $\mathcal{P}_2$ the cone of all increasing concave functions $f(t)$ defined on $(0, \infty)$ such that $f(t) = f(1)t$ $(0 < t \leq 1)$. Show that

$$\mathcal{K}(L_\infty + L_{N_1}) = \mathcal{K}(\ell_1 + \ell_\infty) = \mathcal{P}_2 \tag{5.22}$$

(up to equivalence), where $\mathcal{K}(X_0 + X_1)$ is the set of all functions represented as $f(t) = \mathcal{K}(t, x; X_0, X_1)$ $(x \in X_0 + X_1)$.

If $f \in \mathcal{P}_2$, then the sequence $a_f := (f(i) - f(i-1))_{i=1}^\infty$ belongs to ℓ_∞ and decreases. Moreover, due to (5.21), we have $\mathcal{K}(i, a_f; \ell_1, \ell_\infty) = f(i)$ $(i = 1, 2, \ldots)$. Therefore, taking into account the concavity of these functions and the relation $\mathcal{K}(t, a; \ell_1, \ell_\infty) = t\|a\|_\infty$ $(0 < t \leq 1)$, we obtain that $f(t) \asymp \mathcal{K}(t, a_f; \ell_1, \ell_\infty)$ $(t > 0)$ with constants not depending on $f \in \mathcal{P}_2$. Therefore, $\mathcal{P}_2 \subset \mathcal{K}(\ell_1 + \ell_\infty)$. On the other hand, from the Proposition 5.4, it follows that $\mathcal{K}(\ell_1 + \ell_\infty) \subset \mathcal{K}(L_\infty + L_{N_1})$ and, due to the inequality $\|x\|_{N_1} \leq \|x\|_\infty$, we obtain that $\mathcal{K}(L_\infty + L_{N_1}) \subset \mathcal{P}_2$ (see also Proposition 4.1). Thus, relation (5.22) is proved.

Further, since the couple (L_∞, L_{N_1}) is $\mathcal{K}$-monotone (see Lemma 3.2 and Proposition 4.4), by the Brudnyi–Kruglyak theorem, there exist parameters F_0 and F_1 of the $\mathcal{K}$-method such that

$$X_0 = (L_\infty, L_{N_1})_{F_0}^{\mathcal{K}} \quad \text{and} \quad X_1 = (L_\infty, L_{N_1})_{F_1}^{\mathcal{K}}. \tag{5.23}$$

Therefore, due to the previous corollary, we have

$$q(X_0) = (\ell_1, \ell_\infty)_{F_0}^{\mathcal{K}} \quad \text{and} \quad q(X_1) = (\ell_1, \ell_\infty)_{F_1}^{\mathcal{K}}. \tag{5.24}$$

If $x \in X_0$, then $(\mathcal{K}(2^k, x; L_\infty, L_{N_1}))_k \in F_0$. Due to (5.22), there exists a sequence $a \in \ell_\infty$ such that

$$\mathcal{K}(2^k, a; \ell_1, \ell_\infty) \asymp \mathcal{K}(2^k, x; L_\infty, L_{N_1}) \ \ (k = 0, \pm 1, \pm 2, \ldots). \tag{5.25}$$

Therefore, $(\mathcal{K}(2^k, a; \ell_1, \ell_\infty))_k \in F_0$, i.e., $a \in (\ell_1, \ell_\infty)_{F_0}^{\mathcal{K}}$. Therefore, using (5.24) and the hypothesis of the proposition, we obtain that $a \in (\ell_1, \ell_\infty)_{F_1}^{\mathcal{K}}$, whence, due to (5.23) and (5.25), $x \in X_1$. Thus, $X_0 \subset X_1$. Changing the places of these spaces, we obtain the inverse embedding $X_1 \subset X_0$. Thus, $X_0 = X_1$ as the sets. The norm equivalence in these spaces follows from the fact that every symmetric space on $[0, 1]$ is continuously embedded into L_1 and from the closed graph theorem. □

Remark 5.1. We can show that $q(Z) \in \mathcal{I}(\ell_1, \ell_\infty)$ for any symmetric space Z (compare with Corollary 4.2). Indeed, if B is a linear operator satisfying the $\|B\|_{\ell_1 \to \ell_1} \le 1$ and $\|B\|_{\ell_\infty \to \ell_\infty} \le 1$, then, applying (5.6) and (5.21), we obtain the inequality

$$\|Ba\|_{q(Z)} = \Big\| \sum_{k=1}^{\infty} (Ba)_k^* \chi_{(0,2^{-k+1}]} \Big\|_Z = \Big\| \sum_{k=1}^{\infty} \mathcal{K}(k, Ba; \ell_1, \ell_\infty) \chi_{(2^{-k}, 2^{-k+1}]} \Big\|_Z$$
$$\le \Big\| \sum_{k=1}^{\infty} \mathcal{K}(k, a; \ell_1, \ell_\infty) \chi_{(2^{-k}, 2^{-k+1}]} \Big\|_Z = \Big\| \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{k} a_i^* \Big) \chi_{(2^{-k}, 2^{-k+1}]} \Big\|_Z = \|a\|_{q(Z)}.$$

This implies that the condition that X_0 and X_1 are interpolation spaces with respect to the couple (L_∞, L_{N_1}) in the previous corollary is important. Indeed, let X_0 be a symmetric space not interpolational with respect to the couple (L_∞, L_{N_1}). Then $q(X_0) \in \mathcal{I}(\ell_1, \ell_\infty)$ and therefore, $q(X_0) = (\ell_1, \ell_\infty)_F^{\mathcal{K}}$ for some parameter F. Consider a symmetric space $X_1 := (L_\infty, L_{N_1})_F^{\mathcal{K}}$. By Corollary 5.2, we have $q(X_1) = (\ell_1, \ell_\infty)_F^{\mathcal{K}} = q(X_0)$. At the same time, $X_1 \ne X_0$ because $X_0 \notin \mathcal{I}(L_\infty, L_{N_1})$ and $X_1 \in \mathcal{I}(L_\infty, L_{N_1})$.

Now we are ready to finish the proof of Theorem 5.1.

Proof (*necessity*). As was shown above, it remains to prove that a symmetric space Y defined by (5.14) is interpolation with respect to (L_∞, L_{N_1}).

Let a linear operator A be bounded in spaces L_∞ and L_{N_1}, and let its norm in each of these spaces not exceed 1. First of all, by the definition of the $\mathcal{K}$-functional, we have

$$\mathcal{K}(t, Ax; L_\infty, L_{N_1}) \le \mathcal{K}(t, x; L_\infty, L_{N_1}) \ \ (t > 0).$$

Then, by Lemma 4.4 (see also Remark 4.4), we have

$$\mathcal{K}(t, Ax; L_\infty, L_{N_1}) \asymp t \sup_{0<u\le 2^{1-t}} \big((Ax)^*(u) \log_2^{-1}(2/u) \big) \ge (Ax)^*(2^{1-t}) \ \ (t \ge 1).$$

On the other hand, since φ_1 is equivalent to the fundamental function of the space L_{N_1}, it follows that $\Lambda(\varphi_1) \subset L_{N_1}$. Therefore, by the definition of the operator $\mathcal{L}$ and Lemma 5.3, the following inequality holds:

$$\mathcal{K}(t, x; L_\infty, L_{N_1}) \le C_1 \mathcal{K}(t, x; L_\infty, \Lambda(\varphi_1)) \le C_2 t \int_0^{2^{1-t}} x^*(s)\, d\varphi_1(s)$$
$$\le C_2 \mathcal{L}x(2^{1-t}) \ \ (t \ge 1).$$

From the last relations, it follows that $(Ax)^*(s) \le C\mathcal{L}x(s)$ $(0 < s \le 1)$ for all $x \in \Lambda_2$. Since, due to Lemma 5.1, the operator $\mathcal{L}$ is bounded in Y, it follows that A is bounded in this space as well. $\square$

Comments and references. The idea to identify the subspace of symmetric space generated by the Rademacher system with the corresponding cone of the step function belongs to Rodin and Semenov and appeared in [124]. In that monograph, inequality (5.1) and the assertion of Theorem 5.1 (see Theorem 4) with stronger restrictions for X are proved. This result was used in [124] to prove Theorem 5 which states that any symmetric space is well defined by the subspace generated by the Rademacher system (see the previous chapter). Note that the corresponding statements are proved in that book independently. A weaker variant of Theorem 5.1 (for interpolation spaces with respect to a Banach couple (L_∞, G)) was presented in [32]. The comparison of Proposition 5.5 and Theorem 4.2 shows an interesting analogy between properties of subspaces generated by the Rademacher system and cones of step functions. Methods of the proof of [124, Theorem 5] are used in the proof of Proposition 5.1. The proof of Proposition 5.2 is similar to the one for [58, Theorem 2]. The technique of partial retractors is used.

Chapter 6

RADEMACHER CHAOS IN SYMMETRIC SPACES

Definition 6.1. The set of all functions defined on the segment $I = [0, 1]$ and represented as

$$x(t) = \sum_{1 \le i_1 < i_2 < \ldots < i_d < \infty} a_{i_1, i_2, \ldots, i_d} \, r_{i_1}(t) r_{i_2}(t) \ldots r_{i_d}(t)$$

is called the *Rademacher chaos of order* $d \in \mathbb{N}$.

It is supposed that the convergence of the series on the right-hand side is understood as the convergence a.e. of the respective sequence of "rectangular" sums:

$$S_n(t) = \sum_{1 \le i_1 < i_2 < \ldots < i_d \le n} a_{i_1, i_2, \ldots, i_d} \, r_{i_1}(t) r_{i_2}(t) \ldots r_{i_d}(t) \quad (n \in \mathbb{N}). \tag{6.1}$$

For example, the first-order Rademacher chaos is the set of all convergent series on this system, while the union of chaoses of all orders is contained in the set of functions represented by Walsh series (see [72]).

In the sequel, we consider the second-order chaos because corresponding results for a chaos of arbitrary order are obtained in a similar way (see the Remark at the end of this chapter). Thus, we are interested in the set of real-valued functions $x = x(t)$ in the form

$$x(t) = \sum_{1 \le i < j < \infty} a_{i,j} r_i(t) r_j(t) \quad (t \in I). \tag{6.2}$$

The system $\{r_i r_j\}_{1 \le i < j < \infty}$, which is called the Rademacher chaos, is orthonormal on I (see Corollary 1.2), but the functions of this system are not linearly independent (unlike the usual Rademacher system functions). Moreover, it is not multiplicative, i.e., the integral of a product of pairwise different functions of the system can be non-zero. For example, we have

$$\int_0^1 r_1(t) r_2(t) \, r_1(t) r_3(t) \, r_2(t) r_3(t) \, dt = 1.$$

Nevertheless, properties of the Rademacher chaos are rather similar to properties of families of independent and uniformly bounded functions. The geometrical properties of a subspace "spanned" by this system in an arbitrary symmetric space are our main concern. However, we begin from the a.e. convergence of (6.2). The following result is similar to Theorem 1.1.

Theorem 6.1. *If the sequence* $(a_{i,j})_{1 \le i < j < \infty} \in \ell_2$, *then* (6.2) *converges a.e. on* $I = [0, 1]$.

Proof. We use almost the same reasoning as in the proof of Theorem 1.1. As was mentioned, the system $\{r_i r_j\}_{1 \le i < j < \infty}$ is an orthonormal system in $L_2[0, 1]$. Therefore, since $(a_{i,j})_{1 \le i < j < \infty} \in \ell_2$, it follows that the sequence of partial sums $S_n(t)$ (6.2) defined according to (6.1) is a Cauchy sequence in this space. Hence, $\|S_n - f\|_2 \to 0$ for some function $f \in L_2[0, 1]$. Therefore, for arbitrary $0 \le a < b \le 1$, we have

$$\lim_{n \to \infty} \int_a^b (f(t) - S_n(t))^2 \, dt = 0.$$

Using the Cauchy–Bunyakovskii integral inequality, we see that

$$\lim_{n \to \infty} \int_a^b (f(t) - S_n(t)) \, dt = 0. \tag{6.3}$$

Let $I_k = (a_k, b_k)$ $(k = 1, 2, \ldots)$ be an interval, where $r_k(t)$ is constant. Hence, $S_k(t)$ is constant on the same interval. As every summand of the function $S_n - S_k$ $(n > k)$ contains a factor r_j, where $j > k$, hence,

$$\int_{I_k} (S_n(t) - S_k(t))\, dt = 0,$$

where, due to (6.3), we have

$$\int_{I_k} f(t)\, dt = \int_{I_k} S_k(t)\, dt. \tag{6.4}$$

Assume that E is the set of all binary-irrational points of the segment $[0, 1]$, where the function $F(t) := \int_0^t f(u)\, du$ is differentiable. Since $f \in L_1[0, 1]$, it follows that $m(E) = 1$ (see [110, Theorem 9.4.2]). Moreover, for any point $t_0 \in E$, there exists a sequence of intervals $\{I_k\}_{k=1}^{\infty}$ such that $t_0 \in I_k := (a_k, b_k)$ $(k = 1, 2, \ldots)$ and the function $S_k(t)$ is constant on I_k. Therefore, it follows from (6.4) that

$$S_k(t_0) = \frac{1}{b_k - a_k} \int_{I_k} S_k(t)\, dt = \frac{1}{b_k - a_k} \int_{I_k} f(t)\, dt.$$

Since $b_k - a_k \to 0$, it follows from Lemma 1.1 that the last quantity tends to $F'(t_0) = f(t_0)$ as $k \to \infty$. This means that (6.2) converges at any $t_0 \in E$, i.e., a.e. on $[0, 1]$. $\square$

Consider the following question. In which symmetric spaces do the functions $r_i(t) r_j(t)$ $(1 \le i < j < \infty)$ construct a basis of a subspace generated by them? In other words, when do these functions form a basis sequence? The answer to this question depends on the order in which those functions are considered. As was noted, the sequences $\{r_k\}_{k=1}^{\infty}$ and $\{r_i r_j\}_{1 \le i < j < \infty}$ are subsystems of the Walsh system $\{w_n\}_{n=0}^{\infty}$; therefore, the most suitable order is the one produced by the positions of these functions in $\{w_n\}_{n=0}^{\infty}$. If the latter is considered with the Paley numeration (see [86, Sec. 4.5, p. 150] or [72, Sec. 1.1]), then $w_0 = 1$, $w_{2^k} = r_{k+1}$, $k = 0, 1, \ldots$ According to that enumeration, generate a sequence from the Rademacher chaos:

$$\begin{gathered} \varphi_1 = r_1 r_2,\ \varphi_2 = r_1 r_3,\ \varphi_3 = r_2 r_3, \ldots, \\ \varphi_{(k-2)(k-1)/2+1} = r_1 r_k, \ldots, \varphi_{(k-1)k/2} = r_{k-1} r_k, \ldots \end{gathered} \tag{6.5}$$

Then, as was noted in Sec. 0.4, the system $\{\varphi_i\}_{i=1}^{\infty}$ is a basic sequence in symmetric space X with the following condition: There exists a constant $K > 0$ such that for any natural $m < n$ and $a_i \in \mathbb{R}$ we obtain that

$$\Big\| \sum_{i=1}^{m} a_i \varphi_i \Big\|_X \le K \Big\| \sum_{i=1}^{n} a_i \varphi_i \Big\|_X. \tag{6.6}$$

Show that the Rademacher chaos generates a basic sequence in a quite wide class of symmetric spaces

Theorem 6.2. *The system $\{r_i r_j\}_{1 \le i < j < \infty}$ ordered according to* (6.5) *is a basic sequence in any symmetric interpolation space X on $[0, 1]$ with respect to the couple (L_1, L_∞).*

Proof. In the sequel, the following fact related to properties of the Walsh system $\{w_n\}_{n=0}^{\infty}$ is used (see [72, Sec. 2.1, p. 45]). Let

$$S_p x(t) = \sum_{i=0}^{p} \int_0^1 x(s) w_i(s)\, ds\ w_i(t) \quad (p = 0, 1, 2, \ldots)$$

be operators of partial Fourier–Walsh sums. Then, since

$$\Delta_k^j = \Big((j-1)2^{-k}, j2^{-k}\Big) \quad \text{and} \quad \sigma_k = S_{2^k} \quad (k = 0, 1, 2, \ldots;\ j = 1, 2, \ldots, 2^k),$$

it follows that

$$\sigma_k x(t) = 2^k \int\limits_{\Delta_k^j} x(u)\,du, \quad t \in \Delta_k^j.$$

In other words, the operator σ_k coincides with the averaging operator over the system of dyadic intervals $\{\Delta_k^j\}_{j=1}^{2^k}$. One can easily check (see [92, Sec. 2.3, p. 111]) that this operator is bounded in L_1 and L_∞ and its norm is equal to 1. Therefore, by the condition, operators σ_k are uniformly bounded in X, i.e., for any $B = B(X) > 0$ and all $k = 1, 2, \dots$ and $x \in X$, we have

$$\|\sigma_k x\|_X \le B\,\|x\|_X. \tag{6.7}$$

For arbitrary integers $m < n$ and real a_i, we introduce

$$y(t) = \sum_{i=1}^{n} a_i \varphi_i(t), \quad z(t) = \sum_{i=1}^{m} a_i \varphi_i(t)$$

and prove that inequality (6.6) holds with some constant K depending only on B.

From the fact that the system $\{\varphi_i\}_{i=1}^{\infty}$ is orthonormal and the definition of operators σ_k, for any $0 \le k \le l$, $1 \le p < k+1$, and $1 \le q < l+1$, it follows that $z(t)$ and $y(t)$ are presented as follows

$$z(t) = \sigma_k y(t) + \sum_{j=1}^{p} b_j r_j(t)\, r_{k+1}(t)$$

and

$$y(t) = \sigma_l y(t) + \sum_{j=1}^{q} c_j r_j(t)\, r_{l+1}(t).$$

First, assume that $k < l$. Then

$$\sigma_{k+1} y(t) - \sigma_k y(t) \; = \sum_{j=1}^{k} d_j r_j(t)\, r_{k+1}(t),$$

where $d_j = b_j$ if $1 \le j \le p$. Due to Corollary 1.7 (see inequality (1.25)) and relation (6.7), we have

$$\begin{aligned} \|z\| &\le \|\sigma_k y\| + \Big\| \sum_{j=1}^{p} b_j r_j(t) \Big\| \le B\|y\| + \Big\| \sum_{j=1}^{k} d_j r_j(t) \Big\| \\ &= B\|y\| + \|\sigma_{k+1} y - \sigma_k y\| \le 3B\|y\|; \end{aligned}$$

thus, everything is proved for $k < l$. If $k = l$, then $p \le q$ and $c_j = b_j$ $(1 \le j \le p)$, i.e., $\sigma_{k+1} y = y$; therefore, we have

$$y(t) - \sigma_k y(t) \; = \sum_{j=1}^{q} c_j r_j(t)\, r_{k+1}(t).$$

Thus, the following inequality holds:

$$\begin{aligned} \|z\| &\le \|\sigma_k y\| + \Big\| \sum_{j=1}^{p} b_j r_j(t) \Big\| \le B\|y\| + \Big\| \sum_{j=1}^{q} c_j r_j(t) \Big\| \\ &= B\|y\| + \|y - \sigma_k y\| \le (1 + 2B)\|y\|. \end{aligned}$$

Hence, inequality (6.6) holds. □

Recall (see Corollary 1.7) that the Rademacher system is a symmetric basic sequence in any symmetric space X. Moreover, for arbitrary number sequence $(c_j)_{j=1}^{\infty} \in \ell_2$, the relation

$$\Big\| \sum_{j=1}^{\infty} c_j r_j \Big\|_X = \Big\| \sum_{j=1}^{\infty} c_j^* r_j \Big\|_X$$

holds, where $(c_j^*)_{j=1}^{\infty}$ is a decreasing permutation of the sequence $(|c_j|)_{j=1}^{\infty}$. It is obvious that the Rademacher chaos does not have such good basic properties. In this chapter, we will find the exact conditions for a symmetric space X such that this system (denote it by $\{\varphi_i\}_{i=1}^{\infty}$ again) is symmetric, i.e.,

$$C^{-1} \Big\| \sum_{i=1}^{\infty} a_i \varphi_i \Big\|_X \le \Big\| \sum_{i=1}^{\infty} a_i \varphi_{\pi(i)} \Big\|_X \le C \Big\| \sum_{i=1}^{\infty} a_i \varphi_i \Big\|_X,$$

where the constant $C > 0$ does not depend on the permutation π of the sequence of positive integers. It is interesting that these conditions are necessary and sufficient for the Rademacher chaos to have a weaker property, i.e., to be an unconditional basic sequence.

The latter (see Sec. 0.4) is equivalent to the following fact: there exists a constant K_0 such that for arbitrary $n \in \mathbb{N}$, $\theta_i = \pm 1$ and $a_i \in \mathbb{R}$ $(i = 1, 2, \dots, n)$, the following inequality holds:

$$\Big\| \sum_{i=1}^{n} \theta_i a_i \varphi_i \Big\|_X \le K_0 \Big\| \sum_{i=1}^{n} a_i \varphi_i \Big\|_X.$$

In fact, we prove a stronger assertion: from the unconditionality of the Rademacher chaos in symmetric space X follows the equivalence of the system in X to a canonical basis in ℓ_2.

Let L_{N_1} be the Orlicz space on $[0, 1]$ generated by the function $N_1(t) = e^t - 1$ and $H := (L_{N_1})^\circ$ be a closure of L_∞ in N_1. The main result of this chapter is the following.

Theorem 6.3. *Let X be a symmetric space on $[0, 1]$. Then the following conditions are equivalent to each other*:

(1) *the system $\{r_i r_j\}_{1 \le i < j < \infty}$ is equivalent in X to a canonical basis in space ℓ_2, i.e., for any sequences $(a_{i,j})_{1 \le i < j < \infty} \in \ell_2$, the inequality*

$$C^{-1} \, \|(a_{i,j})\|_2 \le \Big\| \sum_{1 \le i < j < \infty} a_{i,j} r_i r_j \Big\|_X \le C \, \|(a_{i,j})\|_2, \tag{6.8}$$

where the constant $C > 0$ depends only on the space X, is true;

(2) *a continuous embedding $X \supset H$ takes place;*

(3) *the system $\{r_i r_j\}_{1 \le i < j < \infty}$ is an unconditional basic sequence in X.*

The long proof of Theorem 6.3 consists of two main parts. First, we prove the equivalence of (1) $\Leftrightarrow$ (2). Then we prove the implication (3) $\Rightarrow$ (1) (the inverse statement is obvious). We need some auxiliary statements.

Further, we use the decoupling method. Different variants of the method are used to study chaos generated by different systems of independent functions (see comments). Its essence is to pass to the so-called *decoupling chaos*, i.e. (in the case of Rademacher systems), to the set of functions defined on the square $I \times I$ as follows:

$$y(s, t) = \sum_{i,j=1}^{\infty} b_{i,j} r_i(s) r_j(t). \tag{6.9}$$

After the proof of the statement for functions (6.9), we prove it for the "ordinary" chaos, i.e., for the set of functions (6.2).

Recall that for every symmetric space X on $I = [0, 1]$ we can construct a symmetric space $X(I \times I)$ on a square $I \times I$ consisting of all Lebesgue-measurable functions $x = x(s, t)$ on $I \times I$ such that $x^* \in X$ and $\|x\|_{X(I \times I)} = \|x^*\|_X$.

Theorem 6.4. *A system $\{r_i(s)r_j(t)\}_{i,j=1}^{\infty}$ is equivalent in symmetric space $X(I \times I)$ to a canonical basis in ℓ_2 if and only if $X \supset H$.*

Proof. First of all, the equivalence of $\{r_i(s)r_j(t)\}_{i,j=1}^{\infty}$ in $X(I \times I)$ to a canonical basis in ℓ_2 means that

$$\Big\| \sum_{i,j=1}^{\infty} b_{i,j} r_i(s) r_j(t) \Big\|_{X(I\times I)} \asymp \|(b_{i,j})\|_2 \tag{6.10}$$

with constants depending only on the space X.

First, let $X \supset H$ and $y = y(s,t)$ be defined by (6.9) where $b = (b_{i,j})_{i,j=1}^{\infty} \in \ell_2$. If $q \geq 2$, then, applying the Fubini theorem, the Minkovski inequality for the $L_{q/2}$-norm, and (twice) the Khintchine inequality (see Theorem 1.3), we obtain that

$$\begin{aligned} \|y\|_q &= \left(\int\limits_{I\times I} |y(s,t)|^q ds dt \right)^{1/q} = \left(\int\limits_0^1 \int\limits_0^1 |y(s,t)|^q ds dt \right)^{1/q} \\ &\leq \sqrt{q} \left\{ \int\limits_0^1 \left(\sum_{i=1}^{\infty} \Big| \sum_{j=1}^{\infty} b_{i,j} r_j(t) \Big|^2 \right)^{q/2} dt \right\}^{1/q} \\ &\leq \sqrt{q} \left\{ \sum_{i=1}^{\infty} \left(\int\limits_0^1 \Big| \sum_{j=1}^{\infty} b_{i,j} r_j(t) \Big|^q dt \right)^{2/q} \right\}^{1/2} \leq q \|b\|_2. \end{aligned}$$

Finally, we get the inequality $\|y\|_q \leq q\|b\|_2$, which holds for $1 \leq q < 2$. Therefore, in view of the expansion

$$\exp(uz) = \sum_{k=0}^{\infty} \frac{u^k}{k!} z^k \quad (u > 0),$$

the following estimate holds:

$$\int\limits_0^1 \int\limits_0^1 \{\exp(u|y(s,t)|) - 1\}\, ds dt = \sum_{k=1}^{\infty} \frac{u^k}{k!} \int\limits_0^1 \int\limits_0^1 |y(s,t)|^k ds dt \leq \sum_{k=1}^{\infty} \frac{u^k}{k!} k^k \|b\|_2^k.$$

If $u < (2e\|b\|_2)^{-1}$, then, according to (1.14), the sum at the right-hand side of the last inequality does not exceed 1. Therefore, by the definition of the space H, the linear operator

$$Tb(s,t) = \sum_{i,j=1}^{\infty} b_{i,j} r_i(s) r_j(t)$$

is bounded from ℓ_2 to $H(I \times I)$ and, moreover, $\|Tb\|_H = \|y\|_H \leq 2e\|b\|_2$. Since $X \supset H$, we have $\|y\|_X \leq 2Ce\|b\|_2$.

On the other hand, if a function $y(s,t)$ is defined by (6.9), then, due to the embedding $X \subset L_1$, which holds for an arbitrary symmetric space on I, the Khintchine inequality for L_1-norms (see Theorem 1.4 and Remark 1.4), and the Minkowski inequality, we have

$$\begin{aligned} \|y\|_{X(I\times I)} &\geq D^{-1}\|y\|_1 \geq (\sqrt{2}D)^{-1} \int\limits_0^1 \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} b_{i,j} r_j(t) \right)^2 \right]^{1/2} dt \\ &\geq (\sqrt{2}D)^{-1} \left\{ \sum_{i=1}^{\infty} \left[\int\limits_0^1 \Big| \sum_{j=1}^{\infty} b_{i,j} r_j(t) \Big| dt \right]^2 \right\}^{1/2} \geq (2D)^{-1} \|b\|_2. \end{aligned} \tag{6.11}$$

Therefore, relation (6.10) is proved.

To prove the inverse statement, we need an auxiliary lemma. Introduce

$$L(z) = m\{(s,t) \in I \times I :\ \ln(e/s)\ln(e/t) > z\} \ \ (z > 0).$$

Lemma 6.1. *For any $z \geq 1$, the following inequality holds:*

$$\frac{1}{2}\exp(-2\sqrt{z}+2) \leq L(z) \leq 2\exp(-\sqrt{z}+2). \tag{6.12}$$

Proof. If $g_z(u) = u + z/u$ and $h_z(u) = \max(z/u, u)$, then

$$h_z(u) \ \leq\ g_z(u) \ \leq\ 2h_z(u) \ \ (u > 0). \tag{6.13}$$

Changing the variable $u = \ln(e/s)$, we obtain that

$$L(z) = \int_0^1 m\{t \in I :\ \ln(e/t) > z\ln^{-1}(e/s)\}\, ds \ =\ e\int_0^1 \exp\left(-\frac{z}{\ln(e/s)}\right) ds$$
$$= e^2\int_1^\infty \exp(-u - z/u)du = e^2\int_1^\infty \exp(-g_z(u))du.$$

Then, due to (6.13), we have

$$e^{-2}L(z) \leq \int_1^\infty \exp(-h_z(u))du = \int_1^{\sqrt{z}} \exp(-z/u)du + \exp(-\sqrt{z})$$
$$= \exp(-\sqrt{z}) + z\int_{\sqrt{z}}^{z} e^{-u}du/u^2 \leq \exp(-\sqrt{z}) + \sqrt{z}\int_{\sqrt{z}}^{z} e^{-u}du/u \leq 2\exp(-\sqrt{z}).$$

On the contrary, applying (6.13) again, we obtain

$$e^{-2}L(z) \geq \int_1^\infty \exp(-2h_z(u)) \geq \int_{\sqrt{z}}^\infty \exp(-2u)du = 2^{-1}\exp(-2\sqrt{z}),$$

hence, inequality (6.12) is proved. □

Return to the proof of Theorem 6.4 and assume that $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ is equivalent in $X(I \times I)$ to a canonical basis in ℓ_2, i.e., (6.10) holds. If $b_{i,j} = c_i d_j$, then

$$\Big\|\sum_{i,j=1}^\infty b_{i,j}r_i(s)r_j(t)\Big\|_{X(I\times I)} \ \leq\ C\|(b_{i,j})\|_2 \ =\ C\|(c_i)\|_2\|(d_j)\|_2.$$

In particular, for $c_i = d_i = 1/\sqrt{n}\,(1 \leq i \leq n)$ and $c_i = d_i = 0\,(i > n)\,(n = 1, 2, \ldots)$, due to the previous inequality and symmetry of X, we have

$$\|v_n^*(s)v_n^*(t)\|_{X(I\times I)} \ \leq\ C \quad (n = 1, 2, \ldots), \tag{6.14}$$

where $v_n(s) = 1/\sqrt{n}\ \sum_{i=1}^n r_i(s)$.

Applying the central limit theorem in the same way as in the proof of Theorem 3.2, we obtain

$$\lim_{n\to\infty} m\{s \in I :\ v_n^*(s) > z\} \ =\ \Psi(z),$$

where

$$\Psi(z) = \frac{2}{\sqrt{2\pi}} \int\limits_z^\infty \exp(-u^2/2)du \ \ (z > 0).$$

Then, by Lemma 3.1, we have

$$\lim_{n\to\infty} v_n^*(s) \ = \ \Psi^{-1}(s) \ \ (0 < s \le 1).$$

Therefore, using (6.14) and properties of the space X'', we obtain

$$\Psi^{-1}(s)\Psi^{-1}(t) \in X''(I \times I).$$

Since $\Psi(\tau) \ge 2/\sqrt{2\pi}e^{-2\tau^2}$ $(\tau \ge 1)$ (see the proof of Theorem 3.2), the function $\ln^{1/2}(e/s)\ln^{1/2}(e/t)$ belongs to the space $X''(I \times I)$. Therefore, $g(t) = \ln(e/t) \in X''$ due to symmetry of X'' and Lemma 6.1. If $h(t) = 1/\|\chi_{(0,t)}\|_{N_1}$, then

$$h(t) \ = \ N_1^{-1}(1/t) \ = \ \ln(1 + (e-1)/t) \asymp g(t),$$

whence $h \in X''$. For arbitrary $x \in L_\infty$ and all $t \in I$, applying [92, Sec. 2.4, equality (4.39), p. 144], we obtain the inequality

$$x^*(t) \le t^{-1} \int\limits_0^t x^*(s)ds \le t^{-1}\|x\|_{N_1}\|\chi_{(0,t)}\|_{(L_{N_1})'} = \|x\|_{N_1} h(t).$$

Since $(X'')^\circ = X^\circ$ by Lemma 3.3, it follows that

$$\|x\|_X = \|x\|_{X''} \le K\|x\|_{N_1} \ \ (x \in L_\infty), \quad \text{where} \quad K = \|h\|_{X''}.$$

This means that $H = (L_{N_1})^\circ \subset X^\circ \subset X$. Theorem 6.4 is proved. □

To prove Theorem 6.3, we need the following simple statement.

Lemma 6.2. *Let A and B be finite sets of positive integers and $\pi : \mathbb{N} \to \mathbb{N}$ be a one-to-one mapping. Then the absolute values of the functions*

$$y(s,t) = \sum_{i\in A, j\in B} b_{i,j} r_i(s) r_j(t) \ \ (s,t \in I) \quad \text{and} \quad z(s,t) = \sum_{i\in A, j\in B} b_{i,j} r_i(s) r_{\pi(j)}(t) \ \ (s,t \in I)$$

are equimeasurable.

Proof. Indeed, by the Fubini theorem and Proposition 1.4, for any $\tau > 0$, we have

$$\begin{aligned} n_{|z|}(\tau) &= m\{(s,t) \in I \times I : \, |z(s,t)| > \tau\} \\ &= \int\limits_0^1 m\Big\{t \in I : \Big|\sum_{j\in B}\Big(\sum_{i\in A} b_{i,j} r_i(s)\Big) r_{\pi(j)}(t)\Big| > \tau\Big\}\, ds \\ &= \int\limits_0^1 m\Big\{t \in I : \Big|\sum_{j\in B}\Big(\sum_{i\in A} b_{i,j} r_i(s)\Big) r_j(t)\Big| > \tau\Big\}\, ds = n_{|y|}(\tau). \end{aligned}$$

□

Proof of Theorem 6.3 (*the equivalence* (1) ⇔ (2)). The idea of the proof is to pass to the decoupling chaos and apply Theorem 6.4.

First, let $a = (a_{i,j})_{1\le i<j<\infty} \in \ell_2$,

$$x(t) = \sum_{1\le i<j<\infty} a_{i,j} r_i(t) r_j(t) \quad \text{and} \quad x_N(t) = \sum_{1\le i<j\le N} a_{i,j} r_i(t) r_j(t) \ \ (N = 1, 2, \dots).$$

From combinatorial analysis, we see that

$$x_N(t) = 2^{2-N} \sum_{D\subset\{1,\dots,N\}} \Bigl(\sum_{i\in D, j\notin D} a_{i,j} r_i(t) r_j(t) \Bigr), \tag{6.15}$$

where the sum is taken over all sets $D \subset \{1, \dots, N\}$. It is obvious that the absolute values of

$$\sum_{i\in D, j\notin D} a_{i,j} r_i(t) r_j(t)\ (t\in I) \quad \text{and} \quad \sum_{i\in D, j\notin D} a_{i,j} r_i(s) r_j(t)\ (s,t\in I)$$

are equimeasurable. Therefore, if $X \supset H$, then, due to Theorem 6.4, relation (6.15), and the Cauchy–Bunyakovskii inequality, we have

$$\begin{aligned}
\|x_N\|_X &\le 2^{2-N} C \sum_{D\subset\{1,\dots,N\}} \Biggl(\sum_{i\in D, j\notin D} a_{i,j}^2 \Biggr)^{1/2} \\
&\le 2^{2-N} C \Biggl(\sum_{D\subset\{1,\dots,N\}} 1 \Biggr)^{1/2} \Biggl(\sum_{D\subset\{1,\dots,N\}} \sum_{i\in D, j\notin D} a_{i,j}^2 \Biggr)^{1/2} \\
&\le 2^{2-N/2} C \Biggl(2^{N-2} \sum_{1\le i<j\le N} a_{i,j}^2 \Biggr)^{1/2} = 2C \|(a_{i,j})_{1\le i<j\le N}\|_2.
\end{aligned}$$

A similar inequality for the norm of difference of partial sums $\|x_M - x_N\|_X$ $(1 \le N < M < \infty)$ shows that the sequence $\{x_N\}$ is a Cauchy sequence in X; therefore, passing to the limit as $N \to \infty$ in the last relation, we obtain $\|x\|_X \le 2C\|a\|_2$.

One can easily check that the inverse inequality always holds. Indeed, since $X \subset L_1$, applying Proposition 1.3 (see relation (1.19)) for any $N \in \mathbb{N}$, we have the inequality

$$\|(a_{i,j})_{1\le i<j\le N}\|_2 \le B \Bigl\| \sum_{1\le i<j\le N} a_{i,j} r_i r_j \Bigr\|_1 \le B_1 \Bigl\| \sum_{1\le i<j\le N} a_{i,j} r_i r_j \Bigr\|_X.$$

Since the series $\sum_{1\le i<j<\infty} a_{i,j} r_i r_j$ converges in X, passing to the limit as $N \to \infty$ in the last inequality, we have the following relation:

$$\|(a_{i,j})\|_2 \le B_1 \Bigl\| \sum_{1\le i<j<\infty} a_{i,j} r_i r_j \Bigr\|_X. \tag{6.16}$$

Assume the inverse: the function system $\{r_i r_j\}_{1\le i<j<\infty}$ is equivalent in symmetric space X to a canonical basis in ℓ_2. Consider partial sums of series (6.9) presenting the function $y = y(s,t)$:

$$y_N(s,t) = \sum_{i,j=1}^{N} b_{i,j} r_i(s) r_j(t) \ \ (N = 1, 2, \dots).$$

Let the mapping $\pi : \mathbb{N} \to \mathbb{N}$ be one-to-one and

$$\pi(\{1, 2, \dots, N\}) = \{N+1, N+2, \dots, 2N\}.$$

Obviously, the absolute values of

$$y_N'(t) = \sum_{i,j=1}^{N} b_{i,j} r_i(t) r_{\pi(j)}(t)\ (t\in I) \quad \text{and} \quad y_N''(t) = \sum_{i,j=1}^{N} b_{i,j} r_i(s) r_{\pi(j)}(t)\ (s,t\in I)$$

are equimeasurable. Since, by Lemma 6.2, the same is valid for y_N and y''_N, it follows that the absolute values of y_N and y'_N are equimeasurable. Therefore, by the condition, for some $C > 0$, we have

$$\|y_N\|_{X(I\times I)} = \|y'_N\|_X \le C\|(b_{i,j})_{i,j=1}^N\|_2.$$

A similar inequality for norms $\|y_M - y_N\|_{X(I\times I)}$ $(1 \le N < M < \infty)$ shows that the sequence $\{y_N\}$ is a Cauchy sequence in the space $X(I \times I)$. Therefore, passing to the limit as $N \to \infty$ in the previous inequality, we obtain

$$\|y\|_{X(I\times I)} \le C\|(b_{i,j})\|_2.$$

Since the inverse inequality (see (6.11)) holds in any symmetric space, it follows that $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ is equivalent in $X(I \times I)$ to a canonical basis in ℓ_2. Therefore, $X \supset H$ by Theorem 6.4. The equivalence (1) $\Leftrightarrow$ (2) is proved. □

Proof of the implication (3) $\Rightarrow$ (1) is more complicated. First, we obtain it under the additional condition that $X \supset G$, where G is a closure of L_∞ in the Orlicz space L_{N_2} generated by the function $N_2(t) = e^{t^2} - 1$.

Proposition 6.1. *Suppose that X is a symmetric space on $[0,1]$ and $X \supset G$. If a system $\{r_ir_j\}_{1\le i<j<\infty}$ is an unconditional basis sequence in X, then it is equivalent in this space to a canonical basis in ℓ_2.*

To prove this statement, we need a lemma about spaces with mixed norms. Recall the definition (see [84, Sec. 11.1, p. 400] for details).

Definition 6.2. Let X and Y be functional Banach ideal spaces on $I = [0,1]$. The *space $X[Y]$ with mixed norm* consists of all functions $x(s,t)$ measurable on a square $I\times I$ and satisfying two conditions:

(1) for almost every $t \in I$, the function $x(\cdot,t)$ belongs to Y;
(2) the function $\varphi_x(t) = \|x(\cdot,t)\|_Y$ belongs to X.

We set $\|x\|_{X[Y]} = \|\varphi_x\|_X$.

Lemma 6.3. *For arbitrary Orlicz space L_F on I, the following continuous embeddings hold*:

$$L_\infty[L_F] \subset L_F(I\times I) \quad \text{and} \quad L_F(I\times I) \subset L_1[L_F].$$

Proof. If $x = x(s,t) \in L_\infty[L_F]$ and $\alpha = \|x\|_{L_\infty[L_F]}$, then by the definition of the norm in the Orlicz space, we have

$$\int_0^1 F\left(\frac{|x(s,t)|}{\alpha}\right) ds \le 1$$

for almost every $t \in I$. Integrating this inequality and applying the Fubini theorem, we obtain the inequality

$$\int_{I\times I} F\left(\frac{|x(s,t)|}{\alpha}\right) dsdt \le 1.$$

This implies that $x \in L_F(I\times I)$ and $\|x\|_{L_F(I\times I)} \le \|x\|_{L_\infty[L_F]}$. The first embedding is proved. To prove the second one, we pass to associate spaces. If F^* is the function complementary to F, then

$$L_\infty[L_{F^*}] \subset L_{F^*}(I\times I).$$

Then the inverse embedding takes place for the associate spaces:

$$(L_{F^*}(I\times I))' \subset (L_\infty[L_{F^*}])'.$$

Since $(L_F)' = L_{F^*}$, $(F^*)^* = F$ (see [90, Secs. 2 and 14]) and $(X[Y])' = X'[Y']$ (see [44, Theorem 3.12]), we have $L_F(I\times I) \subset L_1[L_F]$ and $\|x\|_{L_1[L_F]} \le C\|x\|_{L_F(I\times I)}$, where $C > 0$. □

Proof of Proposition 6.1. Denote by $\mathbf{r}_{i,j}(u)$ $(1 \le i < j < \infty)$ Rademacher functions indexed in an arbitrary way.

Since G is a subspace of the Orlicz space L_{N_2}, it follows that norms of $L_1[G]$ and $L_1[N_2]$ coincide on functions from $L_\infty(I \times I)$ and norms of $L_\infty[G]$ and $L_\infty[N_2]$ coincide on functions from $L_\infty(I \times I)$. Therefore, since $X \supset G$, it follows from the lemma just proved that the following inequality holds for arbitrary $n \in \mathbb{N}$ and $a_{i,j} \in \mathbb{R}$ $(1 \le i < j \le n)$:

$$\int_0^1 \Big\| \sum_{1 \le i < j \le n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \Big\|_X du = \Big\| \Big\| \sum_{1 \le i < j \le n} a_{i,j} \mathbf{r}_{i,j}(u) r_i(t) r_j(t) \Big\|_{X(t)} \Big\|_{L_1(u)}$$
$$\le C_1 \Big\| \Big\| \sum_{1 \le i < j \le n} a_{i,j} \mathbf{r}_{i,j}(u) r_i(t) r_j(t) \Big\|_{G(u)} \Big\|_{L_\infty(t)}.$$

From the Khintchine inequality for the space G (see Theorem 3.1), it follows that

$$\int_0^1 \Big\| \sum_{1 \le i < j \le n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \Big\|_X du \le C_2 \, \|(a_{i,j})\|_2. \tag{6.17}$$

On the other hand, relation (6.16) shows that the inverse inequality

$$\int_0^1 \Big\| \sum_{1 \le i < j \le n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \Big\|_X du \ge C_3 \, \|(a_{i,j})\|_2 \tag{6.18}$$

always holds. Moreover, if the system $\{r_i r_j\}_{1 \le i < j < \infty}$ is unconditional in X, then, with constants depending only on X, we obtain

$$\Big\| \sum_{1 \le i < j \le n} a_{i,j} r_i r_j \Big\|_X \asymp \int_0^1 \Big\| \sum_{1 \le i < j \le n} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \Big\|_X du$$

for arbitrary $n \in \mathbb{N}$ and $a_{i,j} \in \mathbb{R}$ $(1 \le i < j \le n)$. The statement follows from relations (6.17) and (6.18). $\square$

Combining relations (6.16) and (6.17), we obtain the following statement (for the definition of RUC-systems, see Sec. 0.4).

Corollary 6.1. *If the symmetric space $X \supset G$, then the Rademacher chaos $\{r_i r_j\}_{1 \le i < j < \infty}$ is a RUC-system in X.*

Corollary 6.2. *For any symmetric space X, the following conditions are equivalent to each other*:

(1) $X \supset G$;

(2) $\int_0^1 \Big\| \sum_{1 \le i < j < \infty} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \Big\|_X du \asymp \|(a_{i,j})_{1 \le i < j < \infty}\|_2$.

Proof. The implication (1) $\Rightarrow$ (2) follows from relations (6.17) and (6.18).

If (2) holds, then we take $(a_{i,j})$ such that $a_{1,j} = b_j$ $(j = 2, 3, \dots)$, where $(b_j)_{j=2}^\infty \in \ell_2$ is arbitrary and $a_{i,j} = 0$ $(i \ne 1)$. Then, due to Proposition 1.4 and the condition of the corollary, we obtain

$$\int_0^1 \Big\| \sum_{1 \le i < j < \infty} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \Big\|_X du = \Big\| \sum_{j=2}^\infty b_j r_j \Big\|_X \le C \Big(\sum_{j=2}^\infty b_j^2 \Big)^{1/2}.$$

Since the last inequality holds for any sequence $(b_j)_{j=2}^\infty \in \ell_2$, Theorem 3.2 implies that $X \supset G$. $\square$

Corollary 6.3. *Let the symmetric space $X \supset G$. For any sequence of real numbers $\{a_{i,j}\}_{1\le i<j<\infty}$, there exists a set of signs $\{\theta_{i,j}\}_{1\le i<j<\infty}$, $\theta_{i,j} = \pm 1$, such that*

$$\Big\| \sum_{1\le i<j<\infty} \theta_{i,j} a_{i,j} r_i r_j \Big\|_X \asymp \|(a_{i,j})\|_2.$$

Proof. Due to Corollary 6.2, the following inequality holds:

$$\inf\Big\{\Big\| \sum_{1\le i<j<\infty} \theta_{i,j} a_{i,j} r_i r_j \Big\|_X : \theta_{i,j} = \pm 1\Big\} \le \int_0^1 \Big\| \sum_{1\le i<j<\infty} a_{i,j} \mathbf{r}_{i,j}(u) r_i r_j \Big\|_X du \le C\|(a_{i,j})\|_2.$$

Since the inverse inequality always holds (see (6.16)), we have finished the proof. □

The implication (3) ⇒ (1) of Theorem 6.3 is obtained as a corollary of the particular situation just considered and some geometrical properties of the Rademacher chaos in Marcinkiewicz spaces (in particular, in the space L_∞).

Recall that an ordinary Rademacher system is a Sidon system; more precisely (see Remark 1.5), for any real $a_1, \dots, a_n$, we have

$$\Big\| \sum_{k=1}^n a_k r_k \Big\|_\infty = \sum_{k=1}^n |a_k|.$$

The Rademacher chaos behaves completely differently in L_∞. First, we prove one interesting result for the decoupling chaos $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$.

Proposition 6.2. *Assume that*

$$F(s,t) := \sum_{i,j=1}^\infty a_{i,j} r_i(s) r_j(t) \in L_\infty(I \times I).$$

Then

$$\|(a_{i,j})\|_{l_{4/3}} \le \sqrt{2}\|F\|_\infty. \tag{6.19}$$

Proof. Without loss of generality, we can assume that the sequence $(a_{i,j})_{1\le i<j<\infty}$ is finite, i.e., $a_{i,j} = 0$ for some N and all $j > i \ge N$. The following equality is obvious:

$$\sum_{i,j=1}^\infty |a_{i,j}|^{4/3} = \sum_{i=1}^\infty \left(\sum_{j=1}^\infty |a_{i,j}|^{2/3} |a_{i,j}|^{2/3} \right).$$

Apply Hölder's inequality with indices 3 and 3/2 to the sum with respect to j. Then apply Hölder's inequality with indices 3/2 and 3 to the sum with respect to i. This yields

$$\sum_{i,j=1}^\infty |a_{i,j}|^{4/3} \le \left(\sum_{i=1}^\infty \left(\sum_{j=1}^\infty a_{i,j}^2 \right)^{1/2} \right)^{2/3} \cdot \left(\sum_{i=1}^\infty \left(\sum_{j=1}^\infty |a_{i,j}| \right)^2 \right)^{1/3}.$$

Now, if we estimate the second factor on the right-hand side using the triangle inequality for ℓ_2-norms, then we obtain the inequality

$$\sum_{i,j=1}^\infty |a_{i,j}|^{4/3} \le \left(\sum_{i=1}^\infty \left(\sum_{j=1}^\infty a_{i,j}^2 \right)^{1/2} \right)^{4/3}. \tag{6.20}$$

On the other hand, due to the refined Khintchine L_1-inequality with the exact constant (see Remark 1.4), we have

$$\sum_{i=1}^{\infty}\left(\sum_{i=1}^{\infty}a_{i,j}^2\right)^{1/2} \le \sqrt{2}\sum_{i=1}^{\infty}\int_0^1\Big|\sum_{j=1}^{\infty}a_{i,j}r_j(t)\Big|\,dt = \sqrt{2}\int_0^1\sup_{0<s\le 1}|F(s,t)|\,dt \le \sqrt{2}\|F\|_\infty.$$

Note that (6.19) follows from the last relation and inequality (6.20). □

The following result shows that the exponent 4/3 in inequality (6.19) is exact. We consider "random" sums with respect to the system $\{r_i(s)r_j(t)\}_{1\le i<j<\infty}$ in the space $L_\infty(I\times I)$. For arbitrary integer $n\ge 2$ and $\theta=(\theta_{i,j})_{1\le i<j\le n}, \theta_{i,j}=\pm 1$, introduce a quantity

$$\varphi_n(\theta) = \Big\|\sum_{1\le i<j\le n}\theta_{i,j}r_i(s)r_j(t)\Big\|_{L_\infty(I\times I)}.$$

From the definition of Rademacher functions, it follows that $\max_\theta \varphi_n(\theta)$ is attained at $\theta_{i,j}=1\,(1\le i<j\le n)$ and is equal to $n(n-1)/2$ (the number of summands in the corresponding sum).

Proposition 6.3. *With constants not depending on* $n\ge 2$, *the following relation holds*:

$$\min_\theta \varphi_n(\theta) \asymp 2^{-(n-1)n/2}\sum_\theta \varphi_n(\theta) \asymp n^{3/2}. \tag{6.21}$$

Proof. Let the signs $\theta_{i,j}=\pm 1$ $(1\le i<j\le n)$ be arbitrary. Then, due to the Khintchine L_1-inequality, we have

$$\begin{aligned}\Big\|\sum_{1\le i<j\le n}\theta_{i,j}r_i(s)r_j(t)\Big\|_\infty &= \sup_{0<s\le 1}\sum_{j=2}^{n}\Big|\sum_{i=1}^{j-1}\theta_{i,j}r_i(s)\Big| \ge \sum_{j=2}^{n}\int_0^1\Big|\sum_{i=1}^{j-1}\theta_{i,j}r_i(s)\Big|\,ds\\ &\ge \frac{1}{\sqrt{2}}\sum_{j=2}^{n}\left(\int_0^1\left(\sum_{i=1}^{j-1}\theta_{i,j}r_i(s)\right)^2 ds\right)^{1/2} = \frac{1}{\sqrt{2}}\sum_{j=1}^{n-1}j^{1/2}\ge cn^{3/2},\end{aligned}$$

where $c>0$ does not depend on $n\ge 2$. Therefore, $\min\limits_\theta \varphi_n(\theta) \ge cn^{3/2}$.

To prove the inverse inequality we note that

$$2^{-(n-1)n/2}\sum_\theta \varphi_n(\theta) = \int_0^1\Big\|\sum_{1\le i<j\le n}\mathrm{r}_{i,j}(u)r_i(s)r_j(t)\Big\|_\infty du, \tag{6.22}$$

where $\{\mathrm{r}_{i,j}\}_{1\le i<j\le n}$ are the first $n(n-1)/2$ randomly enumerated Rademacher functions.

Then we apply the distribution theorem for L_∞-norms of polynomials with "random" coefficients (see [82, Theorem 4.1, p. 97]). If the coefficients are Rademacher functions considered on $[0,1]$, then it reads as follows. Let E be a space with measure μ, $\mu(E)<\infty$, and B be a space of real-valued functions, $B\subset L_\infty(E)$. Assume that there exists $\rho>0$ with the following properties: if $f\in B$, then there exists a measurable set $K=K(f)\subset E$ such that $\mu(K)\ge(1/\rho)\mu(E)$ and $|f(x)|\ge 1/2\|f\|_\infty$ $(x\in K)$. Then, for "random" polynomials

$$P(t,x) = \sum_{k=1}^{n}r_k(t)f_k(x) \ \ (f_k\in B)$$

and $z>2$, the following holds:

$$m\left\{t\in[0,1]:\ \|P(t,\cdot)\|_{L_\infty(E_x)}\ge 3\left(\sum_{k=1}^{n}\|f_k\|_\infty^2\ln(2\rho z)\right)^{1/2}\right\}\le\frac{2}{z}. \tag{6.23}$$

Let B_n ($n \in \mathbb{N}$) be a space of functions of the kind

$$f(s,t) = \sum_{1\le i<j\le n} a_{i,j} r_i(s) r_j(t) \ \ (a_{i,j} \in \mathbb{R})$$

defined on $I \times I$, where $I = [0,1]$. Since $|f(s,t)| = \|f\|_\infty$ for any function $f \in B_n$ on any set $K \subset I \times I$ with measure $m(K) \ge 2^{-2n+1}$, it follows from (6.23) that

$$m\Big\{u \in I : \Big\| \sum_{1\le i<j\le n} \mathrm{r}_{i,j}(u) r_i(s) r_j(t) \Big\|_\infty \ge 3\Big[\sum_{1\le i<j\le n} \ln(2^{2n} z)\Big]^{1/2}\Big\} \le \frac{2}{z}$$

for $z > 2$. Since

$$\Big(\sum_{1\le i<j\le n} \ln(2^{2n} z)\Big)^{1/2} \le \Big(\sum_{1\le i<j\le n} 2n \ln z\Big)^{1/2} \le n^{3/2} \ln^{1/2} z,$$

it follows that the distribution of the function

$$X(u) = \Big\| \sum_{1\le i<j\le n} \mathrm{r}_{i,j}(u) r_i(s) r_j(t) \Big\|_\infty \ \ (u \in I)$$

satisfies the estimate

$$m\{u \in I : X(u) \ge 3n^{3/2}\tau\} \le 2e^{-\tau^2} \ \ (\tau \ge 1).$$

Therefore, due to (6.22), the following inequality holds:

$$2^{-(n-1)n/2} \sum_\theta \varphi_n(\theta) = \|X\|_1 = 3n^{3/2} \int_0^\infty m\{u \in I : X(u) \ge 3n^{3/2}\tau\}\, d\tau$$

$$\le 3n^{3/2}\Big(1 + 2\int_1^\infty e^{-\tau^2}\, d\tau\Big) \le 6n^{3/2};$$

this proves the proposition. □

A similar result holds for the "genuine" chaos as well.

Corollary 6.4. *There exists $K > 0$ such that for an arbitrary integer $n \ge 2$ we can find a set $\theta = (\theta_{i,j})_{1\le i<j\le n}$ such that*

$$\Big\| \sum_{1\le i<j\le n} \theta_{i,j} r_i r_j \Big\|_{L_\infty(I)} \le K n^{3/2}. \tag{6.24}$$

Proof. The statement is a corollary of Proposition 6.3 and the inequality

$$\Big\| \sum_{1\le i<j\le n} a_{i,j} r_i r_j \Big\|_{L_\infty(I)} \le \Big\| \sum_{1\le i<j\le n} a_{i,j} r_i(s) r_j(t) \Big\|_{L_\infty(I\times I)},$$

which holds for arbitrary $n \in \mathbb{N}$ and $(a_{i,j})_{1\le i<j\le n}$. □

Recall (see Sec. 0.4) that a sequence of random variables (or measurable functions) $\{f_k\}_{k=1}^\infty$ defined on a probability space is called a Sidon system if for any $n \in \mathbb{N}$ and every polynomial $P(t) = \sum_{k=1}^n a_k f_k(t)$ the following inequalities hold:

$$C^{-1} \sum_{k=1}^n |a_k| \le \|P\|_\infty \le C \sum_{k=1}^n |a_k|,$$

where the constant $C > 0$ does not depend on n and $a_k \in \mathbb{R}$.

From Proposition 6.3 and Corollary 6.4, we obtain the following assertion.

Corollary 6.5. *The sequences* $\{r_i(t)r_j(s)\}_{1\le i<j<\infty}$ *and* $\{r_i(t)r_j(t)\}_{1\le i<j<\infty}$ *are not Sidon systems on* $I\times I$ *and* I *respectively.*

The "probabilistic" proof of Proposition 6.3 does not provide a particular arrangement of signs that the (order) minimum of $\varphi_n(\theta)$ is attained at. Therefore, we provide one more statement for a similar quantity

$$\bar\varphi_n(\theta) \;=\; \Big\|\sum_{i,j=1}^{n}\theta_{i,j}r_i(s)r_j(t)\Big\|_\infty, \quad \theta_{i,j}=\pm1\ (1\le i,j\le n).$$

For simplicity, we consider the case where $n=2^k$.

Proposition 6.4. *Let* $\varepsilon=\{\varepsilon_{i,j}\}_{i,j=1}^{2^k}$ *be the Walsh matrix generated by the first* 2^k *Walsh functions* $(k=0,1,2,\dots)$. *Then* $\bar\varphi_{2^k}(\varepsilon)\le 2^{3k/2}$.

Proof. Recall that the first 2^k Walsh functions are defined by the relations $w_0(t)=1$ and $w_{2^i+j}(t)=r_{i+1}(t)w_j(t)$ $(i=0,1,\dots;\, j=0,\dots,2^i-1)$ (see [86, Sec. 4.5, p. 150] or [72, Sec. 1.3, p. 21]). Since these functions are constant and equal to $+1$ or -1 on the intervals $\Delta_k^i=((i-1)2^{-k}, i2^{-k})$ $(1\le i\le 2^k)$, we can define numbers

$$\varepsilon_{i,j}=\ \operatorname{sign} w_{j-1}(t),\ t\in\Delta_k^i\ \ (i,j=1,2,\dots,2^k),$$

forming a Walsh matrix $\varepsilon=\{\varepsilon_{i,j}\}_{i,j=1}^{2^k}$.

For every $s\in I$, consider a function

$$x_s(u)=\sum_{i=1}^{2^k} r_i(s)\chi_{\Delta_k^i}(u)\ \ (u\in[0,1]).$$

Since $|x_s(u)|=1$ a.e., it follows that $\|x_s\|_2=1$. The Fourier–Walsh coefficients of $x_s(u)$ for $1\le j\le 2^k$ are defined by the relations

$$c_{j-1}(x_s)=\sum_{i=1}^{2^k}\int\limits_{\Delta_k^i} w_{j-1}(u)\,du\, r_i(s) \;=\; 2^{-k}\sum_{i=1}^{2^k}\varepsilon_{i,j}r_i(s).$$

Therefore, due to Hölder's and Bessel's inequalities, we have

$$\begin{aligned}\Big\|\sum_{i,j=1}^{2^k}\varepsilon_{i,j}r_i(s)r_i(t)\Big\|_\infty &= \sup_{0<s\le1}\sum_{j=1}^{2^k}\Big|\sum_{i=1}^{2^k}\varepsilon_{i,j}r_i(s)\Big| = 2^k\sup_{0<s\le1}\sum_{j=1}^{2^k}|c_{j-1}(x_s)|\\ &\le 2^{3k/2}\sup_{0<s\le1}\Big\{\sum_{j=1}^{2^k}(c_{j-1}(x_s))^2\Big\}^{1/2}\le 2^{3k/2}\|x_s\|_2 \;=\; 2^{3k/2}.\end{aligned}$$

This proves the proposition. ∎

To complete the proof of Theorem 6.3, we introduce the following notation. If X is a symmetric space on $[0,1]$, then $\mathcal{R}_{\rm ch}(X)$ is the subspace of X consisting of all functions represented as follows:

$$x(t)=\sum_{1\le i<j<\infty} a_{i,j}r_i(t)r_j(t), \quad\text{where}\quad (a_{i,j})_{1\le i<j<\infty}\in\ell_2.$$

For arbitrary arrangement signs, i.e., for arbitrary sequence $\theta=(\theta_{i,j})_{1\le i<j<\infty}$, $\theta_{i,j}=\pm1$, define the following operator on $\mathcal{R}_{\rm ch}(X)$:

$$T_\theta x(t)=\sum_{1\le i<j<\infty}\theta_{i,j}a_{i,j}r_i(t)r_j(t).$$

Note that if $\{r_i r_j\}_{1\le i<j<\infty}$ is an unconditional sequence in X, then T_θ boundedly acts on $\mathcal{R}_{\mathrm{ch}}(X)$ for any arrangement of signs.

We need one more lemma.

Lemma 6.4. *Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers, $m_k = \frac{1}{2}(n_{k+1}-n_k)(n_{k+1}-n_k-1)$, $t_k = 2^{-n_{k+1}-2}$, and*

$$y_k(t) = \sum_{n_k<i<j\le n_{k+1}} r_i(t)r_j(t) \ \ (k=1,2,\ldots).$$

Let $y(t) = \sum_{k=1}^\infty c_k y_k(t)$ $(t\in[0,1])$, where $c_k>0$ $(k=1,2,\ldots)$. Then

$$y^*(t_k) \ge \sum_{l=1}^k m_l c_l \ \ (k=1,2,\ldots).$$

Proof. If $1\le l\le k$, then, by the definition of Rademacher functions, $y_l(t)=m_l$ for $0<t<2t_k$. Therefore, we have

$$y(t) = \sum_{l=1}^k m_l c_l + \sum_{l=k+1}^\infty c_l y_l(t) \ \ (0<t<2t_k).$$

Moreover, there exists a set $E\subset(0,2t_k)$, $m(E)=t_k$, such that $\sum_{l=k+1}^\infty c_l y_l(t)\ge 0$ for $t\in E$. Therefore, from the previous relation we obtain the inequality

$$y(t) \ge \sum_{l=1}^k c_l m_l \ \ (t\in E).$$

Now, the statement of the lemma follows from the definition of the decreasing rearrangement of a function. □

The following statements show that with a correct choice of signs $\theta_{i,j}$ $(1\le i<j<\infty)$ the image of T_θ defined on $\mathcal{R}_{\mathrm{ch}}(X)$ contains functions with quite large rearrangements.

Proposition 6.5. *There exists a set of signs $\theta=(\theta_{i,j})_{1\le i<j<\infty}$ such that for any $\varepsilon\in(0,1/2)$ there exists a function $x\in\mathcal{R}_{\mathrm{ch}}(L_\infty)$ such that the inequality*

$$(T_\theta x)^*(t) \ge b\,\log_2^{1/2-\varepsilon} 2/t \ \ (0<t\le 2^{-6})$$

holds, where the positive constant b depends only on ε.

Proof. By Corollary 6.4, for any $k\in\mathbb{N}$ there exist signs $\theta_{i,j}=\pm1$ $(2^k<i<j\le 2^{k+1})$ such that the function

$$z_k(t) = \sum_{2^k<i<j\le 2^{k+1}} \theta_{i,j} r_i(t) r_j(t)$$

satisfies the condition

$$\|z_k\|_\infty \le K2^{3k/2}. \tag{6.25}$$

Introduce $x_k(t) = 2^{-(3+2\varepsilon)k/2} z_k(t)$ and

$$x(t) = \sum_{k=1}^\infty x_k(t) = \sum_{1\le i<j<\infty} a_{i,j} r_i(t) r_j(t),$$

where $a_{i,j} = 2^{-(3+2\varepsilon)k/2}\theta_{i,j}$ ($2^k < i < j \le 2^{k+1}$, $k = 1, 2, \dots,$) and $a_{i,j} = 0$ otherwise. From (6.25), it follows that

$$\|x\|_\infty \le K \sum_{k=1}^{\infty} 2^{-\varepsilon k} = \frac{K}{2^\varepsilon - 1},$$

i.e., $x \in \mathcal{R}_{\mathrm{ch}}(L_\infty)$.

Let an arrangement of signs θ consist of found values $\theta_{i,j}$ for $2^k < i < j \le 2^{k+1}$, $k = 1, 2, \dots$ and arbitrary $\theta_{i,j}$ for other pairs (i,j), $i < j$. Then we obtain

$$y(t) = T_\theta x(t) = \sum_{k=1}^{\infty} 2^{-(3+2\varepsilon)k/2}\, y_k(t),$$

where

$$y_k(t) = \sum_{2^k < i < j \le 2^{k+1}} r_i(t) r_j(t) \ \ (k = 1, 2, \dots).$$

Apply Lemma 6.4 for $n_k = 2^k$ and $c_k = 2^{-(3+2\varepsilon)k/2}$. Then $t_k = 2^{-2^{k+1}-2}$ and $m_k \ge 2^{2k-2}$. Therefore, for arbitrary $k = 1, 2, \dots,$ we have

$$y^*(t_k) \ge \frac{1}{4}\sum_{i=1}^{k} 2^{2i} 2^{-(3+2\varepsilon)i/2} \ge 2^{(1/2-\varepsilon)k-2} \ge b' \log_2^{1/2-\varepsilon}(2/t_k),$$

where $b' > 0$ depends only on ε. For every $t \in (0, 2^{-6}]$, there exists an integer k such that $t_{k+1} < t \le t_k$. Then, since $b = 2^{\varepsilon-1/2}b'$, it follows from the previous inequality that

$$y^*(t) \ge y^*(t_k) \ge b' \log_2^{1/2-\varepsilon}(2/t_k) \ \ge\ 2^{\varepsilon-1/2} b' \log_2^{1/2-\varepsilon}(2/t_{k+1}) \ge b \log_2^{1/2-\varepsilon} 2/t.$$

This proves the theorem. $\square$

In the sequel, $M(\varphi_\varepsilon)$ is the Marcinkiewicz space generated by the function $\varphi_\varepsilon(t) = t\log_2^{1/2-\varepsilon} 2/t$ ($\varepsilon < 1/2$).

Corollary 6.6. *Let the symmetric space X on $[0,1]$ have the following property: the operator T_θ is bounded in $\mathcal{R}_{\mathrm{ch}}(X)$ for any arrangement of signs $\theta = (\theta_{i,j})_{1\le i<j<\infty}$. Then*

$$X \supset \bigcup_{\varepsilon\in(0,1/2)} M(\varphi_\varepsilon).$$

Proof. By Theorem 6.5, we find an arrangement of signs such that for any $\varepsilon \in (0, 1/2)$ there exists $x \in \mathcal{R}_{\mathrm{ch}}(L_\infty)$ such that

$$(T_\theta x)^*(t) \ge b \log_2^{1/2-\varepsilon} 2/t \ \ (0 < t \le 2^{-6}).$$

Since $X \supset L_\infty$, it follows from the assumption and the symmetry of this space that $\log_2^{1/2-\varepsilon} 2/t \in X$. Due to Lemma 3.2, this means that $X \supset M(\varphi_\varepsilon)$. Therefore, the corollary is proved. $\square$

Theorem 6.5. *There exists an arrangement of signs of $\theta = (\theta_{i,j})_{1\le i<j<\infty}$ such that for any $\varepsilon \in (-\frac{1}{2}, \frac{1}{2})$ and $\delta \in (0, 1/4 - \varepsilon/2)$ there exists $x \in \mathcal{R}_{\mathrm{ch}}(M(\varphi_\varepsilon))$ such that*

$$(T_\theta x)^*(t) \ge d \log_2^{1/2+\delta}(2/t) \ \ (0 < t \le 2^{-6}), \tag{6.26}$$

where the positive constant d depends only on ε and δ.

Proof. Let $\theta = (\theta_{i,j})_{1\le i<j<\infty}$ and z_k and y_k ($k = 1, 2, \dots$) be defined as in the proof of Proposition 6.5. By Lemma 3.2, the Marcinkiewicz space $M(\varphi_{-1/2}) = M(t\log_2(2/t))$ coincides with the Orlicz space L_{N_1}, $N_1(t) = e^t - 1$. Therefore, due to the proved equivalence (1) $\Leftrightarrow$ (2), we have

$$\|z_k\|_{M(\varphi_{-1/2})} \asymp \Bigg(\sum_{2^k<i<j\le 2^{k+1}} |\theta_{i,j}|^2\Bigg)^{1/2} \asymp 2^k \ \ (k = 1, 2, \dots). \tag{6.27}$$

For arbitrary $u \in (0,1)$, the space $M(\varphi_\varepsilon)$, $\varepsilon = (1-2u)/2$, is a u-type space with respect to a couple $(L_\infty, M(\varphi_{-1/2}))$, i.e., there exists $C > 0$ such that

$$\|x\|_{M(\varphi_\varepsilon)} \le C\,\|x\|_\infty^{1-u}\|x\|_{M(\varphi_{-1/2})}^u \tag{6.28}$$

for all $x \in L_\infty$. Indeed, applying Lemma 3.2 again, we estimate

$$\begin{aligned}\|x\|_{M(\varphi_\varepsilon)} &\le C'\ \sup\{x^*(t)\log_2^{-u}(2/t) : 0 < t \le 1\}\\ &\le C'\ (\sup\{x^*(t) : 0 < t \le 1\})^{1-u}\ \left(\sup\{x^*(t)\ \log_2^{-1}(2/t) : 0 < t \le 1\}\right)^u\\ &\le C\,\|x\|_\infty^{1-u}\ \|x\|_{M(\varphi_{-1/2})}^u.\end{aligned}$$

Finally, from relations (6.25), (6.27), and (6.28), it follows that

$$\|z_k\|_{M(\varphi_\varepsilon)} \le D\,2^{(3-u)k/2}\ \ (k = 1, 2, \ldots). \tag{6.29}$$

For arbitrary $v > 0$, define functions

$$x_k(t) = \ 2^{(u-3-2v)k/2}\,z_k(t) \quad \text{and} \quad x(t) = \sum_{k=1}^\infty x_k(t).$$

From (6.29), it follows that $x \in \mathcal{R}_{\rm ch}(M(\varphi_\varepsilon))$. Moreover, applying Lemma 6.4 with $n_k = 2^k$ and $c_k = 2^{(u-3-2v)k/2}$ to the function

$$y(t) = \ T_\theta x(t) = \sum_{k=1}^\infty 2^{(u-3-2v)k/2}\,y_k(t),$$

we obtain that

$$y^*(t_k) \ge \frac{1}{4}\sum_{i=1}^k 2^{2i}\,2^{(u-2v-3)i/2} \ge 2^{(1+u-2v)k/2-2} \ge d'\log_2^{(1+u-2v)/2}(2/t_k)$$

for every $k = 1, 2, \ldots$, where $t_k = 2^{-2^{k+1}-2}$. Then the proof of Proposition 6.5 implies the inequality

$$y^*(t) \ge d\log_2^{(1+u-2v)/2}(2/t)\ \ (0 < t \le 2^{-6}),$$

where $d > 0$ depends only on u and v. Since, by the condition, $\delta < 1/4 - \varepsilon/2$, we can choose v such that $0 < v < 1/4 - \varepsilon/2 - \delta$. Taking into account that $\varepsilon = (1-2u)/2$, we obtain that $(u-2v)/2 > \delta$. Therefore, relation (6.26) is a corollary of the previous inequality. $\square$

The following assertion is obtained in the same way as Corollary 6.6.

Corollary 6.7. *Let there exist $\varepsilon \in (-1/2, 1/2)$ such that the symmetric space $X \supset M(\varphi_\varepsilon)$. If T_θ is bounded in $\mathcal{R}_{\rm ch}(X)$ for any arrangement of signs $\theta = (\theta_{i,j})_{1\le i<j<\infty}$, then*

$$X\ \supset \bigcup_{0<\delta<1/4-\varepsilon/2} M(\varphi_{-\delta}).$$

Proof of Theorem 6.3 (*implication* (3) $\Rightarrow$ (1)). If the system $\{r_i r_j\}_{1\le i<j<\infty}$ is unconditional in the symmetric space X, then T_θ is bounded in $\mathcal{R}_{\rm ch}(X)$ for any arrangement of signs of θ. Therefore, $X \supset M(\varphi_{1/5})$ by Corollary 6.6. Then, $X \supset M(\varphi_{-1/10})$ by Corollary 6.7. Since $M(\varphi_{-1/10}) \supset G$, we have $X \supset G$. Finally, applying Proposition 6.1, we see that the system $\{r_i r_j\}_{1\le i<j<\infty}$ is equivalent in X to a canonical basis in ℓ_2. $\square$

In Corollary 4.2, it was proved that for any interpolation sequence space E between ℓ_1 and ℓ_2 there exists a functional symmetric space X on $[0,1]$ such that the corresponding sequence space of coefficients with respect to the Rademacher system coincides with E, i.e.,

$$\left\|\sum_{j=1}^{\infty} c_j r_j\right\|_X \asymp \|(c_j)\|_E.$$

If we are dealing with the Rademacher chaos, then the situation is completely different.

Corollary 6.8. *Let E be a Banach ideal sequence space and X be a symmetric space of functions defined on $[0,1]$. Then the relation*

$$\Big\|\sum_{1\le i<j<\infty} a_{i,j} r_i r_j\Big\|_X \asymp \|(a_{i,j})\|_E \tag{6.30}$$

holds if and only if $E=\ell_2$ and $X \supset H$.

Proof. If $E=\ell_2$ and $X \supset H$, then applying Theorem 6.3, we obtain (6.30). Vice versa, it follows from relation (6.30) that$\{r_i r_j\}_{1\le i<j<\infty}$ is an unconditional basic sequence in X. Therefore, $E=\ell_2$ and $X\supset H$ by Theorem 6.3. ☐

Remark 6.1. Similar statements (first of all, Theorems 6.2 and 6.3) hold also for a multiple Rademacher systems $\{r_i(s)r_j(t)\}_{1\le i<j<\infty}$ considered on the square $I\times I$ as well. The proofs are similar to other proofs of this chapter.

Remark 6.2. Theorem 6.3 shows the accuracy of exponential summability of the Rademacher chaos, i.e., the summability for any $\alpha>0$ for the function

$$\exp\Big(\alpha\Big|\sum_{1\le i<j<\infty} b_{i,j} r_i r_j\Big|\Big)$$

such that $\sum\limits_{1\le i<j<\infty} b_{i,j}^2 < \infty$. It is interesting to compare equivalence (1) $\Leftrightarrow$ (2) in this theorem with the statement of Theorem 3.2 for ordinary Rademacher systems. The role of the space H is played by the space G, which is the closure of L_∞ in the Orlicz space L_{N_2}, where $N_2(t)=e^{t^2}-1$.

Remark 6.3. Recall that a Rademacher chaos of an arbitrary order d is the set of all functions represented as follows:

$$x(t)=\sum_{1\le i_1<i_2<\ldots<i_d<\infty} a_{i_1,i_2,\ldots,i_d}\, r_{i_1}(t) r_{i_2}(t)\ldots r_{i_d}(t).$$

Arguing as in the proof of Theorem 6.3, one can easily show that a similar result holds with substitution of H for the space H_d, wich is the closure of L_∞ in the Orlicz space $L_{N_{2/d}}$, generated by the function $N_{2/d}(t)=\exp(t^{2/d})-1$.

To conclude the chapter, we consider the question about compementability of $\mathcal{R}_{\mathrm{ch}}(X)$ in X. The answer is given in terms of the space H (in the same way as in Theorem 3.4 in terms of space G.).

Theorem 6.6. *Let X be a symmetric space on $[0,1]$. The subspace $\mathcal{R}_{\mathrm{ch}}(X)$ of all functions from X represented as (6.2) is complemented in this space if and only if $H\subset X\subset H'$.*

Proof. Regarding the first part of the proof, we make only some remarks because the reasoning is similar to the proof of Theorem 3.4.

First of all, using Theorem 6.3 for spaces X and X', one can easily show that if $H\subset X\subset H'$, then the orthogonal projector

$$Pf(t)=\sum_{1\le i<j<\infty} a_{i,j}(f) r_i(t) r_j(t), \tag{6.31}$$

where

$$a_{i,j}(f) = \int_0^1 f(s) r_i(s) r_j(s)\, ds \ \ (1 \le i < j < \infty), \tag{6.32}$$

is bounded in X.

Further, since the projector P is self-adjoint, we conclude that the embeddings $H \subset X \subset H'$ follow from its boundedness in X. Therefore, it remains to prove that the complementability of $\mathcal{R}_{\rm ch}(X)$ in X implies the boundedness of the projector P in this space.

Recall (see the proof of Theorem 3.4) that an operation

$$t \oplus u = \sum_{i=1}^{\infty} 2^{-i}[(\alpha_i + \beta_i) \bmod 2],$$

where $t = \sum\limits_{i=1}^{\infty} \alpha_i 2^{-i}$ and $u = \sum\limits_{i=1}^{\infty} \beta_i 2^{-i}$ $(\alpha_i, \beta_i = 0, 1)$ are binary decompositions of numbers $t, u \in [0,1]$, transforms the segment $[0,1]$ into a compact Abelian group. For every $u \in [0,1]$, the transformation $w_u(s) = s \oplus u$ preserves the Lebesgue measure on $[0,1]$. Therefore, the operators $T_u f = f \circ w_u$ $(0 \le u \le 1)$ isometrically act in X. One can easily check that the subspace $\mathcal{R}_{\rm ch}(X)$ is invariant with respect to these operators. Therefore, according to the Rudin theorem (see [127, Theorem 5.18]), there exists a bounded projector Q from X on $\mathcal{R}_{\rm ch}(X)$, commuting with all operators T_u $(0 \le u \le 1)$. Show that $Q = P$ (see relations (6.31) and (6.32)).

First, represent the projector Q in the form

$$Qf(t) = \sum_{1 \le i < j < \infty} Q_{i,j}(f) r_i(t) r_j(t), \quad f \in X, \tag{6.33}$$

and show that coordinate functionals $Q_{i,j}$ $(1 \le i < j < \infty)$ are bounded on X. Indeed, due to Proposition 1.3 and the embedding $X \subset L_1$, we get the inequality

$$|Q_{i,j}(f)| \le \|(Q_{i,j}(f))\|_2 \le C' \|Q(f)\|_1 \le C \|Q(f)\|_X \le C \|Q\| \|f\|_X.$$

Moreover, for any $1 \le k < m$, we have

$$Q_{i,j}(r_k r_m) = \begin{cases} 1, & k = i \ \text{ and } \ m = j, \\ 0, & k \ne i \ \text{ or } \ m \ne j. \end{cases} \tag{6.34}$$

Consider the sets

$$\gamma_i = \left\{ u \in [0,1] : u = \sum_{j=1}^{\infty} \alpha_j 2^{-j}, \ \ \alpha_i = 0 \right\}, \quad \overline{\gamma_i} = [0,1] \backslash \gamma_i.$$

One can easily check that

$$r_i(t \oplus u) = \begin{cases} r_i(t), & u \in \gamma_i, \\ -r_i(t), & u \in \overline{\gamma_i}. \end{cases}$$

Therefore, due to the relation $T_u Q = Q T_u$ $(u \in [0,1])$ we have

$$Q_{i,j}(T_u f) = \begin{cases} Q_{i,j}(f), & u \in \gamma_i \cap \gamma_j \ \text{ or } \ u \in \overline{\gamma_i} \cap \overline{\gamma_j} \\ -Q_{i,j}(f), & u \in \gamma_i \cap \overline{\gamma_j} \ \text{ or } \ u \in \overline{\gamma_i} \cap \gamma_j \end{cases}$$

Since

$$m(\gamma_i \cap \gamma_j) = m(\overline{\gamma_i} \cap \gamma_j) = m(\gamma_i \cap \overline{\gamma_j}) = m(\overline{\gamma_i} \cap \overline{\gamma_j}) = \frac{1}{4},$$

it follows that

$$\int_{\gamma_i \cap \gamma_j} Q_{i,j}(T_u f)\, du = \int_{\overline{\gamma}_i \cap \overline{\gamma}_j} Q_{i,j}(T_u f)\, du = \frac{1}{4} Q_{i,j}(f)$$

and

$$\int\limits_{\gamma_i\cap\overline{\gamma}_j} Q_{i,j}(T_u f)\,du = \int\limits_{\overline{\gamma}_i\cap\gamma_j} Q_{i,j}(T_u f)\,du = -\frac{1}{4}Q_{i,j}(f).$$

Note that, due to the boundedness of the coordinate functionals $Q_{i,j}$, we obtain

$$Q_{i,j}(f) = Q_{i,j}\Bigg(\int\limits_{\gamma_i\cap\gamma_j} T_u f\,du + \int\limits_{\overline{\gamma}_i\cap\overline{\gamma}_j} T_u f\,du - \int\limits_{\gamma_i\cap\overline{\gamma}_j} T_u f\,du - \int\limits_{\overline{\gamma}_i\cap\gamma_j} T_u f\,du\Bigg). \tag{6.35}$$

One can easily check that

$$\{s\in[0,1]:\ s = t\oplus u,\quad u\in\gamma_i\} = \begin{cases}\gamma_i, & t\in\gamma_i,\\ \overline{\gamma_i}, & t\in\overline{\gamma_i},\end{cases}$$

$$\{s\in[0,1]:\ s = t\oplus u,\quad u\in\overline{\gamma_i}\} = \begin{cases}\overline{\gamma_i}, & t\in\gamma_i,\\ \gamma_i, & t\in\overline{\gamma_i}.\end{cases}$$

Therefore, since the mapping ω_u preserves the Lebesgue measure on $[0,1]$, it follows that

$$\begin{aligned}\int\limits_{\gamma_i\cap\gamma_j} T_u f(t)\,du = {}& \int\limits_{\gamma_i\cap\gamma_j} f(s)\,ds\cdot\chi_{\gamma_i\cap\gamma_j}(t) + \int\limits_{\overline{\gamma_i}\cap\overline{\gamma_j}} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\overline{\gamma_j}}(t)\\ &+ \int\limits_{\gamma_i\cap\overline{\gamma_j}} f(s)\,ds\cdot\chi_{\gamma_i\cap\overline{\gamma_j}}(t) + \int\limits_{\overline{\gamma_i}\cap\gamma_j} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\gamma_j}(t),\\ \int\limits_{\overline{\gamma}_i\cap\gamma_j} T_u f(t)\,du = {}& \int\limits_{\overline{\gamma}_i\cap\gamma_j} f(s)\,ds\cdot\chi_{\gamma_i\cap\gamma_j}(t) + \int\limits_{\gamma_i\cap\overline{\gamma_j}} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\overline{\gamma_j}}(t)\\ &+ \int\limits_{\overline{\gamma_i}\cap\overline{\gamma_j}} f(s)\,ds\cdot\chi_{\gamma_i\cap\overline{\gamma_j}}(t) + \int\limits_{\gamma_i\cap\gamma_j} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\gamma_j}(t),\\ \int\limits_{\gamma_i\cap\overline{\gamma}_j} T_u f(t)\,du = {}& \int\limits_{\gamma_i\cap\overline{\gamma}_j} f(s)\,ds\cdot\chi_{\gamma_i\cap\gamma_j}(t) + \int\limits_{\overline{\gamma_i}\cap\gamma_j} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\overline{\gamma_j}}(t)\\ &+ \int\limits_{\gamma_i\cap\gamma_j} f(s)\,ds\cdot\chi_{\gamma_i\cap\overline{\gamma_j}}(t) + \int\limits_{\overline{\gamma_i}\cap\overline{\gamma_j}} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\gamma_j}(t)\end{aligned}$$

and

$$\begin{aligned}\int\limits_{\overline{\gamma}_i\cap\overline{\gamma}_j} T_u f(t)\,du = {}& \int\limits_{\overline{\gamma}_i\cap\overline{\gamma}_j} f(s)\,ds\cdot\chi_{\gamma_i\cap\gamma_j}(t) + \int\limits_{\gamma_i\cap\gamma_j} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\overline{\gamma_j}}(t)\\ &+ \int\limits_{\overline{\gamma_i}\cap\gamma_j} f(s)\,ds\cdot\chi_{\gamma_i\cap\overline{\gamma_j}}(t) + \int\limits_{\gamma_i\cap\overline{\gamma_j}} f(s)\,ds\cdot\chi_{\overline{\gamma_i}\cap\gamma_j}(t).\end{aligned}$$

Using these relations, (6.35), and the relation $r_i(t) = \chi_{\gamma_i}(t) - \chi_{\overline{\gamma_i}}(t)$, one can check that

$$Q_{i,j}(f) = Q_{i,j}\Bigg(\int_0^1 f(s)r_i(s)r_j(s)\,ds\cdot r_i(t)r_j(t)\Bigg) = \int_0^1 f(s)r_i(s)r_j(s)\,ds,$$

i.e., $Q_{i,j}(f) = c_{i,j}(f)$ $(1\le 1<j<\infty)$ (see (6.32)). Thus, $Q = P$; the theorem is proved. $\square$

A similar result holds for the decoupling chaos, i.e., for the system $\{r_i(s)r_j(t)\}_{i,j=1}^{\infty}$. Since its proof is quite similar to the proof of the previous theorem, we do not provide it here. Note that one should consider

$$Sf(s,t) = \sum_{i,j=1}^{\infty} \int_0^1 \int_0^1 x(u,v) r_i(u) r_j(v)\, dudv\, r_i(s) r_j(t)$$

instead of the projector P and the transformations $\psi_{u,v}(s,t) = (s \oplus u, t \oplus v)$ $(0 \le u, v \le 1)$ of the square $I \times I$ instead of transformations w_u of the interval $I = [0,1]$.

Theorem 6.7. *The subspace $\mathcal{R}_{\mathrm{dch}}(X)$ of all functions from a symmetric space $X(I \times I)$ presented as* (6.9) *is complemented in $X(I \times I)$ if and only if $H \subset X \subset H'$.*

Remark 6.4. The same methods allow us to consider the complementability question for the subspace $\mathcal{R}_{\mathrm{ch}}^d(X)$ with respect to the Rademacher chaos of arbitrary natural order d. Arguing as above, we can show that $\mathcal{R}_{ch}^d(X)$ is complemented in a symmetric space X if and only if $H_d \subset X \subset (H_d)'$, where (see Remark 6.3) H_d is the closure of L_∞ in the Orlicz space $L_{N_{2/d}}$ generated by the function $N_{2/d}(t) = \exp(t^{2/d}) - 1$.

Comments and References. The origins of chaos theory generated by systems of independent functions can be found in [138]. Now, random polylinear forms introduced there and polynomial chaos corresponding to sequences of independent random variables form a very important area of investigations. Many papers are devoted to them. First of all, we mention [95, Chapter 4], [94, Chapter 6], and references therein: general methods for studying chaos (maximal inequalities, principle of contraction, decoupling method, moment inequalities, convergence questions) have been worked out in detail. On the other hand, the behavior of particular chaoses (first of all, rademacher and gaussian) in L_p-spaces has also been studied. The main aim of Chap. 6 is to study the properties of Rademacher chaos in general symmetric spaces. The content of Chap. 6 is taken from [13, 17].

The condition of Theorem 6.2 that the space X is interpolation with respect to the couple (L_1, L_∞) is not restrictive. All symmetric spaces important for applications (Orlicz, Lorentz, Marcinkiewicz, etc.) have this property (see [92, Sec. 2.4]). Moreover, the given proof shows that this property can be replaced by a weaker one: the uniform boundedness in X for mean operators corresponding to binary decompositions of the segment $[0,1]$.

The main result of the chapter is Theorem 6.3, which is a criterion of unconditionality for Rademacher chaos in symmetric space formulated in terms of embeddings in the same way as Theorem 3.2. The proof of this theorem is based on the idea of decoupling. The main point of that idea is to pass to the so-called decoupling chaos of type (6.9). Many papers are devoted to the decoupling method. In addition to already mentioned monographs, we can mention [60–62, 93], where different variants of decoupling inequalities for moments and distributions of random values with values in Banach spaces are obtained, [83], where applications of decoupling to Rademacher chaos in quasinormed spaces is considered, and [47] and [100] devoted to the application of this method to the geometry of Banach spaces and harmonic analysis. For the behavior of chaos and multiple Rademacher systems in L_p-spaces, see [97, Proposition 2.d.6] and [43, 120]. The latter considers more general subsystems of Walsh functions. These systems are generated by sets of indices of certain combinatorial dimension. In [43] interesting relations between the behavior of those subsystems in spaces L_p and the corresponding variant of the central limit theorem are found. For spaces with mixed norm, see [84, Sec. 11.1]. The result of Lemma 6.3 is well known (see [45, 52, 133]). Examples showing that it holds not for all symmetric spaces can be found in the mentioned works. The notion of the RUC system, which is a natural generalization of the unconditional basis sequence, is introduced in [41]. Nowadays, it is studied intensively, including the theory of symmetric spaces (see [63, 129, 133]). Inequality (6.19) from Proposition 6.2 is called the Littlewood 4/3 inequality; in other words, the sequence $\{r_i(s)r_j(t)\}_{1 \le i < j < \infty}$ is called a Sidon system of order 4/3 (from this point of view, classical

Sidon systems are called Sidon systems of order 1). This has been known for a long time (see [98]). Here we give a proof from [42, Sec. 2.5]. An analog of this inequality for an arbitrary dimension is that the d-multiple Rademacher sequence is a Sidon system of order $2d/(d+1)$ (see [67], [42, Sec. 7.10, Theorem 34]). A result of Corollary 6.5 was proved in a different way in [120, Theorem 4]. It says that the Rademacher chaos is not a Sidon system. Corollary 6.4 gives a quantitative estimate of its nonsidonity, answering the question from [120] about the geometry of a subspace generated by this chaos in the space L_∞. Note that from the decoupling inequalities (see, e.g., [60]), it follows that Rademacher chaoses of order d, as the d-multiple Rademacher sequence, are Sidon systems of order $2d/(d+1)$.

Chapter 7

SUBSPACE $\mathrm{Rad}\, X$

Assume that X is an arbitrary symmetric space on $I = [0,1]$. As above, we denote by $X(I \times I)$ the space of measurable on $I \times I$ functions with the norm

$$\|x\|_{X(I\times I)} = \|x^*\|_X,$$

where $x^*(u)$ is a nonincreasing rearrangement of a function $|x(s,t)|$. In this chapter, we consider properties of the subspace of $X(I \times I)$ generated by Rademacher functions with vector coefficients. We are interested in the set Rad X of all functions $x(s,t) \in X(I \times I)$ representable as

$$x(s,t) = \sum_{i=1}^{\infty} x_i(s) r_i(t) \ \ (x_i \in X) \ \ \text{(the series converges a.e. on } I \times I). \tag{7.1}$$

We begin with the following simple statement.

Proposition 7.1. *For any symmetric space X the set* Rad X *is a closed linear subspace of* $X(I \times I)$.

We need the following lemma. To prove it, we apply Fubini's theorem and Theorems 1.1 and 1.2.

Lemma 7.1. *The series on the right-hand side of* (7.1) *converges a.e. in $I \times I$ if and only if*

$$\sum_{i=1}^{\infty} x_i^2(s) < \infty \ \ \textit{for a.e.} \quad s \in I. \tag{7.2}$$

Proof of Proposition 7.1. Let $\{x^n\} \subset$ Rad X and $\|x^n - x\|_{X(I\times I)} \to 0$ as $n \to \infty$ for some function $x \in X(I \times I)$. Then, by the definition of Rad X, we have

$$x^n(s,t) = \sum_{i=1}^{\infty} x_i^n(s) r_i(t) \ \ (x_i^n \in X).$$

Since any symmetric space is continuously embedded into L_1 (see Sec. 0.2), it follows that $\|x^n - x\|_{L_1(I\times I)} \to 0$ $(n \to \infty)$. Therefore, for any $\varepsilon > 0$ there exists N such that for all $m > n \geq N$ and $k \in \mathbb{N}$, the following inequality holds:

$$\int_0^1 \int_0^1 \Big| \sum_{i=1}^{k} \big(x_i^m(s) - x_i^n(s)\big) r_i(t) \Big| \, ds \, dt \leq \varepsilon.$$

Since, due to the Fubini theorem and the refined Khintchine L_1-inequality (see Remark 1.4), the left-hand side at the last inequality is not less than

$$\frac{1}{\sqrt{2}}\int_0^1\Big(\sum_{i=1}^k\big(x_i^m(s)-x_i^n(s)\big)^2\Big)^{1/2}ds,$$

it follows that

$$\int_0^1\Big(\sum_{i=1}^k\big(x_i^m(s)-x_i^n(s)\big)^2\Big)^{1/2}ds\le\sqrt{2}\varepsilon \tag{7.3}$$

for every $k\in\mathbb{N}$ and all $m>n\ge N$. This shows that the sequence $\{x_i^n\}_{n=1}^\infty$ is a Cauchy sequence in $L_1(I)$ for every $i\in\mathbb{N}$. Therefore, for any $i\in\mathbb{N}$ there exists $x_i\in L_1(I)$ such that $\|x_i^m-x_i\|_{L_1(I)}\to 0$ as $m\to\infty$. Using Fatou's lemma and passing to the limit (first, as $m\to\infty$ with fixed $k\in\mathbb{N}$ and $n\ge N$, and then, as $k\to\infty$) in inequality (7.3), we obtain the inequality

$$\int_0^1\Big(\sum_{i=1}^\infty\big(x_i(s)-x_i^n(s)\big)^2\Big)^{1/2}ds\le\sqrt{2}\varepsilon\ \ (n\ge N). \tag{7.4}$$

Since $x^n\in\operatorname{Rad}X$ $(n\in\mathbb{N})$, it follows from Lemma 7.1 that the sequence

$$f(s,t)=\sum_{i=1}^\infty x_i(s)r_i(t)$$

converges a.e. on $I\times I$. Note that, due to the Parseval identity, the inequality $\|g\|_{L_1(I)}\le\|g\|_{L_2(I)}$, and the Fubini theorem, the left-hand side of (7.4) is not less than

$$\int_0^1\Big(\int_0^1\Big(\sum_{i=1}^\infty\big(x_i(s)-x_i^n(s)\big)r_i(t)\Big)^2dt\Big)^{1/2}ds\ge\int_0^1\int_0^1\Big|\sum_{i=1}^\infty\big(x_i(s)-x_i^n(s)\big)r_i(t)\Big|\,dt\,ds.$$

Therefore, from (7.4), we obtain that

$$\|x^n-f\|_{L_1(I\times I)}\le\sqrt{2}\varepsilon$$

only if $n\ge N$. Therefore, $\|x^n-f\|_{L_1(I\times I)}\to 0$ as $n\to\infty$, whence $x=f\in\operatorname{Rad}X$; the proposition is proved. □

Our next aim is to describe $\operatorname{Rad}X$ as a space of sequences of elements X. Denote by $X(\ell_2)$ the space of sequences $x=\{x_i\}_{i=1}^\infty$, where $x_i\in X$ is such that

$$\sum_{i=1}^\infty x_i^2(s)<\infty\ \text{ for a.e. }\ s\in I\ \text{ and }\ \|x\|_{X(\ell_2)}=\Big\|\Big(\sum_{i=1}^\infty x_i^2\Big)^{\frac12}\Big\|_X<\infty.$$

Theorem 7.1. *Let X be a symmetric space on $I=[0,1]$ with an order semi-continuous norm. Then the following conditions are equivalent to each other*:

(1) *there exists a number $M>0$ such that*

$$M^{-1}\|x\|_{X(\ell_2)}\le\Big\|\sum_{i=1}^\infty x_i(s)r_i(t)\Big\|_{X(I\times I)}\le M\|x\|_{X(\ell_2)} \tag{7.5}$$

for any $x=\{x_k\}_{k=1}^\infty\in X(\ell_2)$;

(2) *the lower Boyd index $\alpha_X>0$.*

To prove this theorem, we need certain properties of operators generated by a tensor product. To be more exact, if $\varphi(t)$ is a function measurable on I, then we assume $T_\varphi x(s,t)=x(s)\varphi(t)$.

Lemma 7.2. (a) *Let $\varphi(t)$ be an unbounded measurable function on I. If an operator T_φ is bounded from X to $X(I \times I)$, then $\alpha_X > 0$.*

(b) *Let $\varphi(t) = \ln^a e/t$, where $a > 0$. If $\alpha_X > 0$, then an operator T_φ is bounded from X to $X(I \times I)$.*

Proof. (a) Assume that $\alpha_X = 0$. This means that $\|\sigma_\tau\|_{X \to X} = 1$ $(0 < \tau \le 1)$; therefore, for any $n = 1, 2, \dots$ there exists a function $x_n \in X$, $\|x_n\| = 1$, such that

$$\|\sigma_{\frac{1}{n}} x_n\| \ge \frac{1}{2}.$$

Since, by assumption, the operator T_φ is bounded from X to $X(I \times I)$ and the functions $x_n(s)\chi_{(0,\frac{1}{n})}(t)$ and $\sigma_{\frac{1}{n}} x_n(s)$ are equimeasurable, it follows that there exists a positive C such that for any $n = 1, 2, \dots$ the following inequality holds:

$$\begin{aligned} C &\ge \|T_\varphi x_n\|_{X(I\times I)} = \|x_n(s)\varphi(t)\|_{X(I\times I)} = \|x_n(s)\varphi^*(t)\|_{X(I\times I)} \\ &\ge \|x_n(s)\varphi^*(n^{-1})\chi_{(0,\frac{1}{n})}(t)\|_{X(I\times I)} = \varphi^*(n^{-1})\|\sigma_{\frac{1}{n}} x_n\|_X \ge \frac{1}{2}\varphi^*(n^{-1}). \end{aligned}$$

This contradicts the unboundedness of $\varphi(t)$. Therefore, the initial assumption is wrong and $\alpha_X > 0$.

(b) By the definition of the lower Boyd index, there exists $C_1 > 0$ such that for all $\tau \in (0,1)$, we have

$$\|\sigma_\tau\|_{X\to X} \le C_1 \tau^{\alpha_X/2}.$$

Therefore, the following inequality holds:

$$\begin{aligned} \|T_\varphi x\|_{X(I\times I)} &= \|T_\varphi x^*\|_{X(I\times I)} \le \Big\| x^*(s) \cdot \sum_{k=1}^{\infty} (k+1)^a \chi_{(e^{-k}, e^{-k+1}]}(t) \Big\|_{X(I\times I)} \\ &\le \sum_{k=1}^{\infty} (k+1)^a \|x^*(s)\cdot \chi_{(e^{-k}, e^{-k+1}]}(t)\|_{X(I\times I)} \le \sum_{k=1}^{\infty} (k+1)^a \|\sigma_{e^{-k+1}} x\|_X \\ &\le C_1 \sum_{k=1}^{\infty} e^{(1-k)\alpha_X/2} (k+1)^a \|x\|_X = C_a \|x\|_X. \end{aligned}$$

The lemma is proved. $\square$

Proof of Theorem 7.1. First of all, if $\{x_k\}_{k=1}^\infty \subset X$, then $\{x_k(s) r_k(t)\}_{k=1}^\infty$ is an unconditional basic sequence in $X(I \times I)$ with constant 1, i.e.,

$$\Big\| \sum_{i=1}^{n} \theta_i x_i(s) r_i(t) \Big\|_{X(I\times I)} = \Big\| \sum_{i=1}^{n} x_i(s) r_i(t) \Big\|_{X(I\times I)} \tag{7.6}$$

for any $n \in \mathbb{N}$ and $\theta_i = \pm 1$ $(i = 1, 2, \dots, n)$. Indeed, by the Fubini theorem and Proposition 1.4, for any $\tau > 0$, the followinf relation holds:

$$\begin{aligned} m\Big\{(s,t) \in I \times I : \Big|\sum_{i=1}^{\infty} \theta_i x_i(s) r_i(t)\Big| > \tau\Big\} &= \int_0^1 m\Big\{t \in I : \Big|\sum_{i=1}^{\infty} \theta_i x_i(s) r_i(t)\Big| > \tau\Big\}\, ds \\ = \int_0^1 m\Big\{t \in I : \Big|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big| > \tau\Big\}\, ds &= m\Big\{(s,t) \in I \times I : \Big|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big| > \tau\Big\}, \end{aligned}$$

now, (7.6) follows from the definition of symmetric space. This implies that the sequence $\{x_k(s) r_k(t)\}_{k=1}^\infty$ is monotone, i.e.,

$$\Big\| \sum_{i=1}^{n} x_i(s) r_i(t) \Big\|_{X(I\times I)} \le \Big\| \sum_{i=1}^{n+1} x_i(s) r_i(t) \Big\|_{X(I\times I)} \quad (n = 1, 2, \dots). \tag{7.7}$$

Further, due to (7.6) and the Minkovski and Khintchine inequalities (see Theorem 1.4 and Remark 1.4), for any $n = 1, 2, \dots$, we have

$$\Big\|\sum_{i=1}^{n} x_i(s) r_i(t)\Big\|_{X(I\times I)} = \int_0^1 \Big\|\sum_{i=1}^{n} x_i(s) r_i(t) r_i(u)\Big\|_{X(I\times I)}\, du$$

$$\geq \Big\|\int_0^1 \Big|\sum_{i=1}^{n} x_i(s) r_i(t) r_i(u)\Big|\, du\Big\|_{X(I\times I)} \geq \frac{1}{\sqrt{2}}\Big\|\Big(\sum_{i=1}^{n} x_i(s)^2\Big)^{1/2}\Big\|_{X(I\times I)} = \frac{1}{\sqrt{2}}\Big\|\Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}\Big\|_X.$$

Now, if

$$\sum_{i=1}^{\infty} x_i(s) r_i(t) \in X(I\times I),$$

then due to the order semi-continuity of the norm in X and inequality (7.7), we have

$$\Big\|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big\|_{X(I\times I)} = \lim_{n\to\infty}\Big\|\sum_{i=1}^{n} x_i(s) r_i(t)\Big\|_{X(I\times I)}.$$

Therefore, passing to the limit as $n \to \infty$ in the previous inequality, we obtain the inequality

$$\Big\|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big\|_{X(I\times I)} \geq \frac{1}{\sqrt{2}}\|\{x_k\}_{k=1}^{\infty}\|_{X(\ell_2)}. \tag{7.8}$$

Note that inequality (7.8) holds in any symmetric space X.

Now, we pass to the proof of the theorem. First, assume that $\alpha_X > 0$. If $\{x_k\}_{k=1}^{\infty} \in X(\ell_2)$, then

$$f := \Big(\sum_{i=1}^{\infty} x_i^2\Big)^{1/2} \in X \quad \text{and} \quad \|f\|_X = \|\{x_k\}\|_{X(\ell_2)}.$$

By Corollary 1.5, for almost every $s \in I$, the following inequality holds:

$$m\Big\{t \in I : \Big|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big| > \tau\Big\} \leq C \exp\left(-\frac{\tau^2}{2f(s)^2}\right).$$

Since

$$\exp\left(-\frac{\tau^2}{2f(s)^2}\right) \leq m\{t \in I :\, 2f(s)\ln^{1/2} e/t > \tau\} \ \ (0 < s \leq 1),$$

it follows that, integrating with respect to $s \in I$, we obtain the inequality

$$m\Big\{(s,t) \in I\times I : \Big|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big| > \tau\Big\} \leq Cm\{(s,t) \in I\times I :\, 2f(s)\ln^{1/2} e/t > \tau\},$$

whence

$$\Big\|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big\|_{X(I\times I)} \leq 2C\|T_\varphi f\|_{X(I\times I)},$$

where $\varphi(t) = \ln^{1/2} e/t$. Finally, applying Assumption (b) of Lemma 7.2, we come to the inequality

$$\Big\|\sum_{i=1}^{\infty} x_i(s) r_i(t)\Big\|_{X(I\times I)} \leq M\|f\|_X = M\|\{x_k\}\|_{X(\ell_2)}.$$

Prove the inverse statement. Assume that relation (7.5) (exactly, its right-hand side) holds. Let

$$\Psi(u) = \frac{2}{\sqrt{2\pi}}\int_u^{\infty} e^{-\frac{v^2}{2}}\, dv, \ \ 0 < u < \infty.$$

Denote by $\varphi(t)$ the function inverse to $\Psi(u)$. The function $\varphi(t)$ is defined on $[0,1]$ and infinitely increases as $t \to +0$. Applying the same arguing as in the proof of Theorem 3.2, we show that the operator T_φ is bounded from X to $X(I \times I)$; then we use statement (a) of Lemma 7.2.

Let $z_n(t) = n^{-\frac{1}{2}} \sum\limits_{i=1}^{n} r_i(t)$. Due to the central limit theorem (see [46, Theorem 7.2]), the following relation holds for all $u > 0$:

$$\lim_{n\to\infty} m\{t \in [0,1] : |z_n(t)| > u\} = \Psi(u).$$

By Lemma 3.1, it follows that $\lim\limits_{n\to\infty} z_n^*(t) = \varphi(t)$ $(0 < t \le 1)$.

For arbitrary $x \in X''$ and $n = 1, 2, \ldots$, we define $y_k^n = \frac{1}{\sqrt{n}} x$ $(1 \le k \le n)$ and $y_k^n = 0$ $(k > n)$. It is obvious that $y^n := \{y_k^n\}_{k=1}^{\infty} \in X''(\ell_2)$ and $\|y^n\|_{X''(\ell_2)} = \|x\|_{X''}$.

Assume that $x^N(s) = \min\{N, |x(s)|\}$ $(N = 1, 2, \ldots)$. Then $x^N \uparrow |x|$ and $x^N \in L_\infty \subset X$. Since the norm of the space X is order semi-continuous, it follows that X is isometrically embedded into the space X'', which is maximal (has Fatou's property). Therefore, by assumption, the following inequality holds:

$$\|x^N(s) z_n^*(t)\|_{X(I\times I)} = \Big\| \sum_{i=1}^{n} \frac{1}{\sqrt{n}} x^N(s) r_i(t) \Big\|_{X(I\times I)} \le M \|y^n\|_{X''(\ell_2)} = M\|x\|_{X''}.$$

Passing to the limit as $N \to \infty$ and $n \to \infty$ in the last relation, we obtain the inequality

$$\|T_\varphi x\|_{X''(I\times I)} = \|x(s)\varphi(t)\|_{X''(I\times I)} \le M\|x\|_{X''} \ \ (x \in X'').$$

Since $\alpha_X = \alpha_{X''}$ (see [92, Theorem 2.4.11]), it follows from Lemma 7.2 that $\alpha_X > 0$. Theorem 7.1 is proved. $\square$

Consider the question about complementability of Rad X in symmetric space $X(I \times I)$. Recall that a similar problem is considered in Chap. 3 (see Theorem 3.4) for the case of scalar coefficients. It was proved that the subspace $\mathcal{R}(X)$ is complemented in symmetric space X if and only if we have the embeddings $G \subset X \subset G'$, where G is the separable part of the Orlicz space L_{N_2}, $N_2(t) = e^{t^2} - 1$.

Show that in the case of Rad X, a more restrictive condition about the non-triviality of Boyd indices is sufficient and necessary.

Theorem 7.2. *Let X be a symmetric space on $I = [0,1]$ with an order semi-continuous norm. Then the following conditions are equivalent to each other*:

(1) *the subspace* Rad X *is complemented in* $X(I \times I)$;
(2) *the Boyd indices of the space X are non-trivial, i.e.,*

$$0 < \alpha_X \le \beta_X < 1.$$

Properties of the following orthogonal projector are used in the proof of the theorem:

$$Pf(s,t) = \sum_{i=1}^{\infty} \int_0^1 f(s,v) r_i(v) dv \cdot r_i(t). \tag{7.9}$$

Proposition 7.2. *The projector P is bounded in a symmetric space $X(I \times I)$ if and only if the Boyd indices of the space X are non-trivial. If the latter condition is satisfied, then the image of P coincides with* Rad X.

Proof. First, assume that $0 < \alpha_X \le \beta_X < 1$. If $f(s,t) \in L_p(I \times I)$ with some $1 < p < \infty$, then, by the Fubini theorem, $f(s,\cdot) \in L_p(I)$ for almost every $s \in I$. Therefore (see the proof of Theorem 3.4), the projector P is defined on $\cup_{1<p<\infty} L_p(I \times I)$ (the series from the right-hand side of (7.9) converges a.e.

on $I \times I$) and, for every $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that for almost every $s \in I$, we have

$$\int_0^1 |Pf(s,t)|^p\,dt \le C_p \int_0^1 |f(s,t)|^p\,dt.$$

Integrating the last inequality over $I = [0,1]$, we obtain that P is bounded in $L_p(I \times I)$ for all $1 < p < \infty$. Therefore, due to the Boyd interpolational theorem (see [97, Theorem 2.b.11] and [92, Theorem 2.6.12]) the sequence from the right-hand side of (7.9) converges a.e. on $I \times I$ for any $f \in X$ and the operator P is bounded in X. Therefore, the image of P is contained in Rad X. Vice versa, if $x(s,t) \in$ Rad X, i.e., this function can be presented as (7.1), then $\sum_{i=1}^{\infty} x_i^2(s) < \infty$ for almost every $s \in [0,1]$ by Lemma 7.1. Since this implies that $Px = x$, it follows that the image of P coincides with Rad X.

Prove the inverse statement. Let P be bounded in $X(I \times I)$ and $A := \|P\|_{X(I\times I)\to X(I\times I)}$. Introduce

$$< f,g >:= \int_0^1 \int_0^1 f(s,t)g(s,t)\,ds\,dt \ \ (f \in X, g \in X').$$

Since the norm of X is order semi-continuous, it follows that X is isometrically embedded into the second associate space X''. Moreover, one can easily check that $< Pf, g >=< f, Pg >$. Therefore, we have

$$\begin{aligned} A &= \sup_{\|f\|_{X(I\times I)}\le 1} \|Pf\|_{X''(I\times I)} = \sup_{\|f\|_{X(I\times I)}\le 1}\ \sup_{\|g\|_{X'(I\times I)}\le 1} < Pf, g > \\ &= \sup_{\|g\|_{X'(I\times I)}\le 1}\ \sup_{\|f\|_{X(I\times I)}\le 1} < f, Pg > \ = \sup_{\|g\|_{X'(I\times I)}\le 1} \|Pg\|_{X'(I\times I)}. \end{aligned}$$

Thus, P is bounded in $X'(I \times I)$ and

$$\|P\|_{X'(I\times I)\to X'(I\times I)} = A. \tag{7.10}$$

If f belongs to Rad X and can be represented in the form $f(s,t) = \sum_{i=1}^{\infty} x_i(s) r_i(t)$ $(x_i \in X)$, then by the Fubini theorem, the following relation holds for any $g \in X'(I \times I)$:

$$\begin{aligned} < f,g > &= \int_0^1 \int_0^1 \sum_{i=1}^{\infty} x_i(s)r_i(t)\cdot g(s,t)\,ds\,dt = \int_0^1 \sum_{i=1}^{\infty} x_i(s) \int_0^1 g(s,t)r_i(t)\,dt\,ds \\ &= \int_0^1 \int_0^1 \sum_{i=1}^{\infty} x_i(s)r_i(t) \cdot \sum_{i=1}^{\infty} \int_0^1 g(s,v)r_i(v)\,dv\ r_i(t)\,dt\,ds \ = < f, Pg > . \end{aligned}$$

Therefore, taking into account the order semi-continuity of the norm of X and relation (7.10), we obtain the inequality

$$\begin{aligned} \|f\|_{X(I\times I)} &= \sup\{< f,g > \,:\, \|g\|_{X'(I\times I)} \le 1\} = \sup\{< f,Pg > \,:\, \|g\|_{X'(I\times I)} \le 1\} \\ &\le \sup\{< f,h > \,:\, h \in \text{Rad } X, \|h\|_{X'(I\times I)} \le A\}. \end{aligned} \tag{7.11}$$

For arbitrary $x = \{x_i\} \in X(\ell_2)$ and $y = \{y_i\} \in X'(l_2)$, we assume that

$$(x,y) := \sum_{i=1}^{\infty} \int_0^1 x_i(s)y_i(s)ds.$$

Then, since $f \in \mathrm{Rad}\, X$ and $g \in \mathrm{Rad}\, X$, i.e.,

$$f(s,t) = \sum_{i=1}^{\infty} x_i(s) r_i(t) \quad \text{and} \quad g(s,t) = \sum_{i=1}^{\infty} y_i(s) r_i(t),$$

where $x_i \in X$ and $y_i \in X$, we have $< f, g >= (x,y)$. Moreover, due to the Cauchy–Bunyakovskii inequality we see that $|(x,y)| \le \|x\|_{X(\ell_2)} \|y\|_{X'(l_2)}$. Therefore, $X'(l_2) \subset (X(\ell_2))^*$ and $\|y\|_{(X(\ell_2))^*} \le \|y\|_{X'(l_2)}$. As we see from relation (7.8), the following inequality is true:

$$\Big\| \sum_{i=1}^{\infty} y_i(s) r_i(t) \Big\|_{X'(I \times I)} \ge \frac{1}{\sqrt{2}} \|y\|_{X'(l_2)}.$$

Taking into account (7.11), we obtain the inequality

$$\Big\| \sum_{i=1}^{\infty} x_i(s) r_i(t) \Big\|_{X(I \times I)} \le \sup\{(x,y) : y \in X'(l_2), \|y\|_{X'(l_2)} \le \sqrt{2} A\}$$
$$\le \sup\{(x,y) : y \in X'(l_2),\ \|y\|_{(X(\ell_2))^*} \le \sqrt{2} A\} \le \sqrt{2} A \|x\|_{X(\ell_2)}.$$

From this inequality and Theorem 7.1, it follows that $\alpha_X > 0$.

According to (7.10), the projector P is bounded in $X'(I \times I)$. Therefore, $\alpha_{X'} > 0$. Since the norm of X is order semi-continuous, it follows that $\beta_X = 1 - \alpha_{X'}$ (see [92, Theorem 2.4.11]). Therefore, $\beta_X < 1$. The proposition is proved. □

Proof of Theorem 7.2. Due to Proposition 7.2, it suffices to show that from the compplementability of Rad X in $X(I \times I)$, it follows that the projector P is bounded in this space.

Let $t = \sum\limits_{i=1}^{\infty} \alpha_i 2^{-i}$ and $u = \sum\limits_{i=1}^{\infty} \beta_i 2^{-i}$ $(\alpha_i, \beta_i = 0, 1)$ be the binary expansion of numbers $t, u \in [0,1]$. As in the proof of Theorem 3.4, we define the following operation:

$$t \oplus u = \sum_{i=1}^{\infty} 2^{-i} [(\alpha_i + \beta_i) \bmod 2].$$

This operation transforms the segment $[0,1]$ into a compact Abelian group. For $(s,t) \in I \times I$ and $u \in I$, we assume that $w_u(s,t) = (s, t \oplus u)$. Transformations w_u preserve the Lebesgue measure on $I \times I$. Therefore, operators T_u $(0 \le u \le 1)$ defined by the relations

$$T_u f(s,t) = f \circ w_u(s,t) = f(s, t \oplus u)$$

isometrically act in $X(I \times I)$. Moreover, one can easily check that Rad X is invariant with respect to those operators. Therefore, according to the Rudin theorem (see [127, Theorem 5.18]), there exists a projector Q bounded from $X(I \times I)$ to Rad X and commuting with all operators T_u $(0 \le u \le 1)$. Show that $Q = P$.

Represent the projector Q in the form

$$Qf(s,t) = \sum_{i=1}^{\infty} Q_i f(s) r_i(t), \tag{7.12}$$

where Q_i are linear operators from $X(I \times I)$ to X. One can easily check that

$$Q_i(x(s) r_j(t)) = \begin{cases} x(s), & i = j, \\ 0, & i \ne j. \end{cases} \tag{7.13}$$

Consider sets

$$\gamma_i = \{u \in [0,1] : u = \sum_{i=1}^{\infty} \alpha_j 2^{-j},\ \ \alpha_i = 0\} \quad \text{and} \quad \overline{\gamma_i} = [0,1] \backslash \gamma_i.$$

One can easily check that

$$r_i(t+u) = \begin{cases} r_i(t), & u \in \gamma_i, \\ -r_i(t), & u \in \overline{\gamma_i}. \end{cases}$$

From this relation, the relation $w_u^2 = I$, and the commutativity of $T_u Q = Q T_u$ $(0 \le u \le 1)$, it follows that

$$Q_i(f \circ w_u) = \begin{cases} Q_i f, & u \in \gamma_i, \\ -Q_i f, & u \in \overline{\gamma_i}. \end{cases}$$

Therefore, since $m(\gamma_i) = m(\overline{\gamma_i}) = \dfrac{1}{2}$, we have

$$\int\limits_{\gamma_i} Q_i(f \circ w_u)\,du = \frac{1}{2} Q_i f, \quad \int\limits_{\overline{\gamma_i}} Q_i(f \circ w_u)\,du = -\frac{1}{2} Q_i f. \tag{7.14}$$

Then, due to (7.6) and (7.12), we see that

$$\|Q_i f\|_X \le \|Qf\|_{X(I\times I)} \le \|Q\| \|f\|_X,$$

i.e., the operators Q_i $(i = 1, 2, \dots)$ are bounded. Therefore, in (7.14), operators Q_i can be taken out of the sign of the integral, whence

$$Q_i f = Q\left(\int\limits_{\gamma_i} (f \circ w_u) du - \int\limits_{\overline{\gamma_i}} (f \circ w_u) du \right). \tag{7.15}$$

For every $u \in [0,1]$, the mapping $t \mapsto t \oplus u$ preserves the Lebesgue measure on $[0,1]$. Moreover, for any $t \in [0,1]$, we have

$$\{v \in [0,1] :\ v = t \oplus u, \ \ u \in \gamma_i\} = \begin{cases} \gamma_i, & t \in \gamma_i, \\ \overline{\gamma_i}, & t \in \overline{\gamma_i}, \end{cases}$$

$$\{v \in [0,1] :\ v = t \oplus u, \ \ u \in \overline{\gamma_i}\} = \begin{cases} \overline{\gamma_i}, & t \in \gamma_i, \\ \gamma_i, & t \in \overline{\gamma_i}. \end{cases}$$

Therefore,

$$\int\limits_{\gamma_i} (f \circ w_u)(s,t)\,du = \int\limits_{\gamma_i} f(s,v)\,dv \cdot \chi_{\gamma_i}(t) + \int\limits_{\overline{\gamma_i}} f(s,v)\,dv \cdot \chi_{\overline{\gamma_i}}(t)$$

and

$$\int\limits_{\overline{\gamma_i}} (f \circ w_u)(s,t)\,du = \int\limits_{\overline{\gamma_i}} f(s,v)\,dv \cdot \chi_{\gamma_i}(t) + \int\limits_{\gamma_i} f(s,v)\,dv \cdot \chi_{\overline{\gamma_i}}(t).$$

One can check that $r_i = \chi_{\gamma_i} - \chi_{\overline{\gamma_i}}$ $(i = 1, 2, \dots)$. Therefore, from the last two relations, it follows that

$$\int\limits_{\gamma_i} (f \circ w_u)(s,t)\,du - \int\limits_{\overline{\gamma_i}} (f \circ w_u)(s,t)\,du = \int\limits_0^1 f(s,v) r_i(v)\,dv \cdot r_i(t).$$

This and (7.13) imply that

$$Q_i f(s) = \int\limits_0^1 f(s,v) r_i(v)\,dv \quad (i = 1, 2, \dots).$$

Relations (7.9) and (7.12) allow us to conclude that $Q = P$. Therefore, the projector P is bounded. $\square$

Comments and References. The introduced definitions of the spaces Rad X and $X(\ell_2)$ are different from the ones used in [97, Section 2.d], but they coincide if X is separable. The sufficiency of the condition $\alpha_X > 0$ for (7.5) to hold (see Theorem 7.1) is proved in [97, Proposition 2.d.1]; the necessity is proved in [26]. Tensor products and operators generated by it (see Lemma 7.2) were studied in many works, see, e.g., [5, 7, 10, 30, 31, 103–106, 112]. If the coefficients of the series in (7.5) are independent functions, this relation holds if and only if X has the Kruglov property (see [25]). Theorem 7.2 is proved in [26].

Chapter 8

SYSTEMS EQUIVALENT IN DISTRIBUTION TO THE RADEMACHER SYSTEM

The main aim of this chapter is to compare arbitrary systems of functions (random variables) with the "model" Rademacher system. To clarify the meaning of the word "comparison," we introduce the following definitions.

Definition 8.1. A system of random variables $\{f_i\}_{i=1}^{\infty}$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ is *majorized in distribution* by a system of random variables $\{g_i\}_{i=1}^{\infty}$ defined on a probability space $(\Omega', \Sigma', \mathbb{P}')$ (we write $\{f_i\} \overset{\mathbb{P}}{\prec} \{g_i\}$) if there exists $C > 0$ such that for arbitrary $m \in \mathbb{N}$, $a_i \in \mathbb{R}$ $(i = 1, 2, \dots, m)$, and $z > 0$, the following inequality holds:

$$\mathbb{P}\left\{\omega \in \Omega : \left|\sum_{i=1}^{m} a_i f_i(\omega)\right| > z\right\} \le C\mathbb{P}'\left\{\omega' \in \Omega' : \left|\sum_{i=1}^{m} a_i g_i(\omega')\right| > C^{-1} z\right\}.$$

Definition 8.2. We say that systems of random variables $\{f_i\}$ and $\{g_i\}$ are *equivalent in distribution* (we write $\{f_i\} \overset{\mathbb{P}}{\sim} \{g_i\}$) if $\{f_i\} \overset{\mathbb{P}}{\prec} \{g_i\}$ and $\{g_i\} \overset{\mathbb{P}}{\prec} \{f_i\}$. In other words, $\{f_i\} \overset{\mathbb{P}}{\sim} \{g_i\}$ if there exists $C > 0$ such that for arbitrary $m \in \mathbb{N}$, $a_i \in \mathbb{R}$ $(i = 1, 2, \dots, m)$, and $z > 0$, the following inequality holds:

$$C^{-1}\mathbb{P}\left\{\omega \in \Omega : \left|\sum_{i=1}^{m} a_i f_i(\omega)\right| > Cz\right\} \le \mathbb{P}'\left\{\omega' \in \Omega' : \left|\sum_{i=1}^{m} a_i g_i(\omega')\right| > z\right\}$$
$$\le C\mathbb{P}\left\{\omega \in \Omega : \left|\sum_{i=1}^{m} a_i f_i(\omega)\right| > C^{-1} z\right\}.$$

We search for conditions such that a general system of random variables is majorized by the Rademacher system considered on $[0, 1]$ or is equivalent to it. An important tool is the notion of the Peetre $\mathcal{K}$-functional of an element $x \in E_0 + E_1$ with respect to the Banach couple (E_0, E_1). As above, we use the notation

$$\kappa_a(t) = \mathcal{K}(t, a; \ell_1, \ell_2),$$

where $a = (a_i)_{i=1}^{\infty} \in \ell_2$ and $t > 0$.

First we consider the majorizability property and prove that for uniformly bounded systems, the relation $\{f_i\} \overset{\mathbb{P}}{\prec} \{r_i\}$ is equivalent to the following "strong" summability condition for series with respect to the system $\{f_i\}_{i=1}^{\infty}$: if a sequence of coefficients $a = (a_i)_{i=1}^{\infty}$ belongs to ℓ_2, then the function

$$f(\omega) = \sum_{i=1}^{\infty} a_i f_i(\omega) \in L_{N_2}(\Omega),$$

where L_{N_2} is the Orlicz space generated by the function $N_2(t) = e^{t^2} - 1$.

Recall that, by Theorem 4.1, Rademacher sums

$$x(t) = \sum_{i=1}^{\infty} a_i r_i(t), \quad \text{where} \quad a = (a_i)_{i=1}^{\infty} \in \ell_2, \tag{8.1}$$

satisfy the relation

$$\mathcal{K}(t, x; L_\infty, G) \asymp \kappa_a(t) \ \ (t > 0),$$

where G is the closure of L_∞ in the Orlicz space L_{N_2}. Obtain a similar relation for the $\mathcal{K}$-functional of the couple (L_p, L_∞) $(1 \le p < \infty)$.

Proposition 8.1. *If* $1 \le p < \infty$, *then the following relation holds for functions* $x(t)$ *in the form of* (8.1):

$$\mathcal{K}(u, x; L_p, L_\infty) \asymp u \cdot \kappa_a(\ln^{1/2} e/u), \tag{8.2}$$

where the constants do not depend on $a = (a_i)_{i=1}^{\infty}$ *and on* $0 < u \le 1$.

Proof. First of all, we recall that, by Corollary 2.2,

$$x^*(u) \le C_1 \kappa_a(\ln^{1/2} e/u) \ \ (0 < u \le 1)$$

and

$$x^*(\beta u) \ge C_2^{-1} \kappa_a(\ln^{1/2} e/u) \ \ (0 < u \le 1),$$

where $\beta \in (0,1)$, $C_1 > 0$, and $C_2 > 0$. This implies that

$$\int_0^u x^*(s)^p \, ds \asymp \int_0^u \kappa_a(\ln^{1/2} e/s)^p \, ds \ \ (0 < u \le 1), \tag{8.3}$$

where the constants depend only on p.

Show that

$$\int_0^u \kappa_a(\ln^{1/2} e/s)^p \, ds \asymp u \cdot \kappa_a(\ln^{1/2} e/u)^p \ \ (0 < u \le 1). \tag{8.4}$$

First, let $u = e^{-k}$, $k \in \mathbb{N}$. Due to the concavity of $\kappa_a(t)$, the following relation holds:

$$\int_0^{e^{-k}} \kappa_a(\ln^{1/2} e/s)^p \, ds = \sum_{i=k}^{\infty} \int_{e^{-i-1}}^{e^{-i}} \kappa_a(\ln^{1/2} e/s)^p \, ds \asymp \sum_{i=k}^{\infty} \kappa_a(i^{1/2})^p \, e^{-i}.$$

Obviously, the last sum is greater than $\kappa_a(k^{1/2})^p \, e^{-k}$. On the other hand, due to the concavity of $\kappa_a(t)$, it is less than

$$\frac{\kappa_a(k^{1/2})^p}{k^{p/2}} \sum_{i=k}^{\infty} i^{p/2} e^{-i} \le e \frac{\kappa_a(k^{1/2})^p}{k^{p/2}} \int_k^{\infty} x^{p/2} \cdot e^{-x} \, dx.$$

Integrating by parts, we obtain

$$\begin{aligned} \int_k^{\infty} x^{p/2} \cdot e^{-x} \, dx &= k^{p/2} e^{-k} + \frac{p}{2} \int_k^{\infty} x^{p/2-1} \cdot e^{-x} \, dx \\ &\le k^{p/2} e^{-k} + \frac{p}{2k} \int_k^{\infty} x^{p/2} \cdot e^{-x} \, dx. \end{aligned}$$

Therefore, for $k > p/2$, we have

$$k^{p/2}e^{-k} < \int\limits_k^\infty x^{p/2}\cdot e^{-x}\,dx \le \frac{2k}{2k-p}k^{p/2}e^{-k},$$

whence

$$\lim_{k\to\infty} k^{-p/2}e^k \int\limits_k^\infty x^{p/2}\cdot e^{-x}\,dx = 1.$$

Therefore, due to the above results, the following inequality holds:

$$\sum_{i=k}^\infty \kappa_a(i^{1/2})^p\, e^{-i} \le C\cdot\kappa_a(k^{1/2})^p\, e^{-k}$$

where $C > 0$ depends only on p. Therefore, relation (8.4) holds for $u = e^{-k}$, $k \in \mathbb{N}$.

If $u \in (0,1]$ is arbitrary, then we choose $k \in \mathbb{N}$ such that $e^{-k} < u \le e^{-k+1}$. Then, taking into account the concavity of the function $\kappa_a(t)$ and using the case already considered we obtain:

$$\int\limits_0^u \kappa_a(\ln^{1/2} e/s)^p\,ds \asymp \int\limits_0^{e^{-k}} \kappa_a(\ln^{1/2} e/s)^p\,ds \asymp e^{-k}\kappa_a(\ln^{1/2}(e^{k+1}))^p \asymp u\kappa_a(\ln^{1/2} e/u)^p,$$

therefore, relation (8.4) is proved.

From Sec. 0.5 (see also [40, Theorem 5.2.1] for details) we know that the $\mathcal{K}$-functional with respect to the Banach couple (L_p, L_∞), is defined by the relation

$$\mathcal{K}(u,x;L_p,L_\infty) \asymp \Big(\int\limits_0^{u^p} x^*(s)^p\,ds\Big)^{1/p}.$$

Therefore, using (8.3), (8.4), and the concavity of the function $\kappa_a(t)$ we obtain (8.2). The proposition is proved. $\square$

Applying the statement just proved for $p = 1$ and the representation for the $\mathcal{K}$-functional $\kappa_a(t)$ from Lemma 2.2, we obtain the following assertion:

Corollary 8.1. *There exist a constant not depending on $a = (a_i)_{i=1}^\infty \in \ell_2$ and $0 < u \le 1$, such that*

$$\int\limits_0^u \Big(\sum_{i=1}^\infty a_i r_i(t)\Big)^* dt \;\asymp\; u\left\{\sum_{i=1}^{[\ln e/u]} a_i^* + \ln^{1/2} e/u\Big[\sum_{i=[\ln e/u]+1}^\infty (a_i^*)^2\Big]^{1/2}\right\}.$$

Remark 8.1. Relation (8.2) can be extended for all $u > 0$. Indeed, due to the definition of the $\mathcal{K}$-functional, the inequalities $\|x\|_p \le \|x\|_\infty$ and $\|a\|_2 \le \|a\|_1$, and Theorem 1.4 for the arbitrary sum $x(t)$ in the form of (8.1) and $u \ge 1$, we have the relation

$$\mathcal{K}(u,x;L_p,L_\infty) \asymp \|x\|_p \asymp \|a\|_2 \asymp u\cdot\kappa_a(1/u) \asymp u\kappa_a(\ln^{1/2}(1+1/u^2))$$

(the constants of the second equivalence depend on p). Combining this with relation (8.2), we obtain the relation

$$\mathcal{K}(u,x;L_p,L_\infty) \asymp u\kappa_a(\ln^{1/2}(1+1/u^2))\;\;(u>0). \tag{8.5}$$

Moreover, since for an arbitrary Banach couple (E_0,E_1), we have

$$\mathcal{K}(u,x;E_0,E_1) = u\mathcal{K}(1/u,x;E_1,E_0)\;\;(u>0),$$

it follows from (8.5) that

$$\mathcal{K}(u,x;L_\infty,L_p) \asymp \kappa_a(\ln^{1/2}(1+u^2))\;\;(u>0). \tag{8.6}$$

Here the constants at (8.5) and (8.6) depend only on p.

The properties of the function $\kappa_a(t)$ from Corollary 2.1 are important for the proofs of the following two theorems. Denote $\bar{\kappa}_a(t) := \kappa_a(\sqrt{t})$.

Theorem 8.1. *For an arbitrary system of random variables* $\{f_i\}_{i=1}^{\infty}$ *defined on a probability space* $(\Omega, \Sigma, \mathbb{P})$, *the following conditions are equivalent to each other*:

(a) *there exists* $C > 0$ *not depending on* $m \in \mathbb{N}$, $a = (a_i)_{i=1}^m$, *and* $t \in [0,1]$ *such that*

$$\int_0^t \Big(\sum_{i=1}^m a_i f_i\Big)^*(s)\, ds \le C\, t\, \mathcal{K}(\ln^{1/2} e/t, a; \ell_1, \ell_2); \tag{8.7}$$

(b) *there exists* $B > 0$ *not depending on* $m \in \mathbb{N}$, $a = (a_i)_{i=1}^m$, *and* $p \ge 1$ *such that*

$$\Big\| \sum_{i=1}^m a_i f_i \Big\|_p \le B\, \mathcal{K}(\sqrt{p}, a; \ell_1, \ell_2); \tag{8.8}$$

(c) $\{f_i\} \overset{\mathbb{P}}{\prec} \{r_i\}$.

Moreover, if a system $\{f_i\}_{i=1}^{\infty}$ *is uniformly bounded, i.e.,* $|f_i(\omega)| \le D$, $\omega \in \Omega, i = 1, 2, \dots,$ *then each of Conditions* (a)–(c) *is equivalent to the following condition*:

(d) *for an arbitrary sequence* $a = (a_i)_{i=1}^{\infty} \in \ell_2$, *the function* $f = \sum_{i=1}^{\infty} a_i f_i$ *belongs to the Orlicz space* L_{N_2}.

Proof. (a) $\Rightarrow$ (b). Due to Proposition 8.1 and the representation of the $\mathcal{K}$-functional in the Banach couple (L_1, L_∞) (see Sec. 0.5), relations (8.7) can be written as follows:

$$\mathcal{K}\Big(u, \sum_{i=1}^m a_i f_i; L_1, L_\infty\Big) \le C' \mathcal{K}\Big(u, \sum_{i=1}^m a_i r_i; L_1, L_\infty\Big) \quad (0 < u \le 1). \tag{8.9}$$

Since L_p $(1 \le p \le \infty)$ is interpolation with respect to the $\mathcal{K}$-monotone couple (L_1, L_∞) (see [92, Sec. 2.4] or [40]), relation (8.8) is a corollary of the last inequality and Theorem 2.1.

(b) $\Rightarrow$ (c). First of all, $\bar{\kappa}_a(t) = \mathcal{K}(\sqrt{t}, a; \ell_1, \ell_2)$ $(t \ge 1)$. Therefore, if $f = \sum_{i=1}^m a_i f_i$, then, due to the Chebyshev inequality and relation (8.8), the inequality

$$\mathbb{P}\{|f(\omega)| \ge \lambda \bar{\kappa}_a(t)\} \le \lambda^{-t} \left[\frac{\|f\|_t}{\bar{\kappa}_a(t)}\right]^t \le \left(\frac{B}{\lambda}\right)^t$$

holds for arbitrary $t \ge 1$ and $\lambda > 0$. In particular, if $\lambda = Be$, then

$$\mathbb{P}\{|f(\omega)| \ge Be\bar{\kappa}_a(t)\} \le e^{-t} \quad (t \ge 1)$$

or

$$\mathbb{P}\{|f(\omega)| \ge Be\bar{\kappa}_a(t)\} \le e^{1-t} \quad (t > 0). \tag{8.10}$$

Further, for an arbitrary $z > 0$, by Corollary 2.1, we find $t' > 0$ such that $0 < \bar{\kappa}_a(t') \le (Be)^{-1} z$. Then it follows from (8.10) that

$$\mathbb{P}\{|f(\omega)| > z\} \le \mathbb{P}\{|f(\omega)| \ge Be\bar{\kappa}_a(t')\} \le e^{1-t'}.$$

Therefore, if

$$F(s) = \sup\{t > 0 : \bar{\kappa}_a(t) \le s\} \quad (s > 0), \tag{8.11}$$

then

$$\mathbb{P}\{|f(\omega)| > z\} \le e^{1 - F(C_1^{-1} z)} \quad (z > 0) \tag{8.12}$$

where $C_1 = Be$. Note that from (8.8) and (2.2), it follows that

$$\|f_n\|_\infty = \lim_{t\to+\infty} \|f_n\|_t \le B \;\; (n \in \mathbb{N});$$

therefore, $\|f\|_\infty \le B\|a\|_1$. Therefore,

$$\mathbb{P}\{|f(\omega)| > z\} = 0, \quad \text{if} \quad z \ge B\|a\|_1. \tag{8.13}$$

Recall that, by Theorem 2.1, there exists constant γ from $(0, 1)$ not depending on $t \in [1, \infty)$ and $a = (a_n)_{n=1}^\infty \in \ell_2$ such that

$$\gamma \bar{\kappa}_a(t) \le \Big\| \sum_{n=1}^m a_n r_n \Big\|_t \le \bar{\kappa}_a(t). \tag{8.14}$$

Now, if $r(x) = \sum\limits_{n=1}^m a_n r_n(x)$ $(0 \le x \le 1)$, then due to the Paley–Zygmund inequality (2.14), for all $t \ge 1$, the inequality

$$\mathbb{P}\{|r(x)| \ge \frac{\gamma}{2}\bar{\kappa}_a(t)\} \ge m\{|r(x)|^t \ge 2^{-t}\|r\|_t^t\}$$
$$\ge (1-2^{-t})^2 \frac{\|r\|_t^{2t}}{\|r\|_{2t}^{2t}} \ge \gamma^{2t}(1-2^{-t})^2 \left\{\frac{\bar{\kappa}_a(t)}{\bar{\kappa}_a(2t)}\right\}^{2t} \ge \Big(\frac{\gamma}{4}\Big)^{2t}$$

holds (the concavity of the $\mathcal{K}$-functional and the fact that $1 - 2^{-t} \le 2^{-t}$ $(t \ge 1)$ are used). Since $\bar{\kappa}_a(t)$ increases with respect to t, it follows that

$$m\Big\{|r(x))| \ge \frac{\gamma}{2}\bar{\kappa}_a(t)\Big\} \ge \Big(\frac{\gamma}{4}\Big)^{2t+2}$$

for all $t > 0$. If $0 < z < \gamma\|a\|_1/16$, then, applying Corollary 2.1 again, we find $t'' > 0$ such that $\bar{\kappa}_a(t'') \ge 4z\gamma^{-1}$. Then, from the previous inequality, it follows that

$$m\{|r(x)| > z\} \ge m\Big\{|r(x)| \ge \frac{\gamma}{2}\bar{\kappa}_a(t'')\Big\} \ge \Big(\frac{\gamma}{4}\Big)^{2t''+2}.$$

Therefore, if

$$G(s) = \inf\{t > 0 : \bar{\kappa}_a(t) \ge s\} \;\; (s > 0),$$

then

$$m\{|r(x)| > z\} \ge \Big(\frac{\gamma}{4}\Big)^{2G(4z\gamma^{-1})+2},$$

whence

$$m\{|r(x)| > z\} \ge C_2^{-1} e^{-C_3 G(4z\gamma^{-1})} \quad \text{if} \quad 0 < z < \frac{\gamma}{16}\|a\|_1, \tag{8.15}$$

where $C_2 = (4/\gamma)^2$, and $C_3 = 2\ln(4/\gamma)$.

One can easily see that $F(s) \le G(s)$ $(s > 0)$, where the functional $F(s)$ is defined by (8.11). At the same time, for $C_4 = 4(2C_3)^{\frac{1}{2}}\gamma^{-1}$, the following inequality holds:

$$C_3 G(4z\gamma^{-1}) \le F(C_4 z) \;\; (z > 0).$$

Indeed, due to the concavity of the $\mathcal{K}$-functional, the definitions of $G(s)$ and $\bar{\kappa}_a(t)$, and the quality $C_3 \ge 1$, we obtain the inequality

$$\bar{\kappa}_a(C_3 G(4z\gamma^{-1})) \le (2C_3)^{\frac{1}{2}} \bar{\kappa}_a(2^{-1}G(4z\gamma^{-1})) \le 4(2C_3)^{\frac{1}{2}}\gamma^{-1} z = C_4 z$$

it remains to use the definition of $F(s)$. Therefore, (8.15) can be rewritten as follows:

$$m\{|r(x)| > z\} \ge C_2^{-1} e^{-F(C_4 z)} \quad \text{if} \quad 0 < z < \frac{\gamma}{16}\|a\|_1. \tag{8.16}$$

Further, let

$$A := \max\Big(C_2 e\,,\, C_1 C_4\,,\, \frac{16B}{\gamma}\Big). \tag{8.17}$$

Note that, since the constant γ is universal, it follows that A actually depends not on the system $\{f_n\}_{n=1}^{\infty}$, but only on B, which is the constant of relation (8.8).

If $z/A < \gamma\|a\|_1/16$, then from (8.12), (8.16), and (8.17) and the increasing of $F(s)$, it follows that

$$\mathbb{P}\{|f(\omega)| > z\} \le e^{1-F(C_1^{-1}z)} \le e^{1-F(C_4 z/A)} \le C_2 e\, m\{|r(x)| > z/A\} \le A\, m\{|r(x)| > z/A\}. \qquad (8.18)$$

In the case where $z/A \ge \gamma\|a\|_1/16$, from (8.17), we obtain the inequality $z \ge B\|a\|_1$. Therefore, due to (8.13), $\mathbb{P}\{|f(\omega)| > z\} = 0$ and (8.18) holds. Inequality (8.18) means that $\{f_n\} \overset{\mathbb{P}}{\prec} \{r_n\}$, and the implication $(b) \Rightarrow (c)$ is proved.

(c) $\Rightarrow$ (a). By the condition, the following inequality holds:

$$\mathbb{P}\left\{\omega \in \Omega : \left|\sum_{i=1}^{m} a_i f_i(\omega)\right| > z\right\} \le Cm\left\{x \in [0,1] : \left|\sum_{i=1}^{m} a_i r_i(x)\right| > C^{-1}z\right\} \ (z > 0).$$

Due to the boundedness of the dilation operator in symmetric space, this implies inequality (8.9) and, therefore, inequality (8.7) equivalent to it. Therefore, the proof of the equivalence of Conditions (a)–(c) is finished.

(c) $\Rightarrow$ (d). By Theorem 3.1, $r(x) = \sum\limits_{i=1}^{\infty} a_i r_i(x) \in L_{N_2}$ if $a = (a_i)_{i=1}^{\infty} \in \ell_2$. Since by assumption, $n_{|f|}(z) \le C n_{|r|}(z/C)$ $(z > 0)$ for $f(\omega) = \sum\limits_{i=1}^{\infty} a_i f_i(\omega)$, it follows from the symmetry of the space L_{N_2} that $f \in L_{N_2}$ and (d) is proved. Note that the uniform boundedness of functions of the system $\{f_i\}_{i=1}^{\infty}$ is not used here (it follows from Condition (c)).

(d) $\Rightarrow$ (a). First of all, check that the linear operator $T(a_i) = \sum\limits_{i=1}^{\infty} a_i f_i$ boundedly acts from ℓ_2 to L_{N_2}. Consider the corresponding maximal operator

$$M(a_i)(\omega) = \sup_{n=1,2,\dots} |T_n(a_i)(\omega)| \quad \text{where} \quad T_n(a_i)(\omega) = \sum_{i=1}^{n} a_i f_i(\omega).$$

Obviously, each of operators T_n is linear and bounded from ℓ_2 to the space $S = S(\Omega, \Sigma, \mathbb{P})$ of all measurable a.e. finite functions on Ω. This space is considered with the topology of convergence with respect to the probability $\mathbb{P}$. Moreover, by the condition, the inequality $M(a_i)(\omega) < \infty$ holds a.e. on Ω for any sequence $(a_i) \in \ell_2$. Therefore, by the Banach theorem of the continuity of maximal operators (see [2, Theorem 2.7]), M boundedly acts from ℓ_2 to S. Then T has this property a fortiori. It remains to apply the closed graph theorem. Let $(a_i^m) \to (a_i)$ in ℓ_2 and $T(a_i^m) \to g$ in L_{N_2} as $m \to \infty$. Since L_{N_2} is continuously embedded into S, it follows that $T(a_i^m) \to g$ in S as well. On the other hand, as we proved above, $T(a_i^m) \to T(a_i)$ in S. Therefore, $g = T(a_i)$ and, by the closed graph theorem, the operator T is bounded from ℓ_2 to L_{N_2}.

Recall that by, Lemma 3.2, the Orlicz space L_{N_2} coincides with the Marcinkiewicz space $M(\varphi)$, where $\varphi(t) = t\ln^{1/2}(e/t)$. Therefore, the boundedness of T is equivalent to the inequality

$$\int_0^t \left(\sum_{i=1}^{\infty} a_i f_i\right)^*(s)\, ds \ \le\ C_5 t \ln^{1/2}(e/t)\|a\|_2,$$

where $C_5 > 0$ does not depend on $a = (a_i)_{i=1}^{\infty} \in \ell_2$ and $t \in [0,1]$. Now, let $a_k^* = |a_{i_k}|$ $(k = 1, 2, \dots)$ be a non-increasing permutation of the sequence $(|a_i|)_{i=1}^{\infty}$. Due to the previous inequality and assumptions of the theorem, for arbitrary $j \in \mathbb{N}$, we have

$$\begin{aligned}
\int_0^{e^{-j}} \Big(\sum_{i=1}^{\infty} a_i f_i\Big)^*(s)\,ds \;&\le \int_0^{e^{-j}} \Big(\sum_{k=1}^{j} a_{i_k} f_{i_k}\Big)^*(s)\,ds + \int_0^{e^{-j}} \Big(\sum_{k=j+1}^{\infty} a_{i_k} f_{i_k}\Big)^*(s)\,ds \\
&\le e^{-j} D \sum_{k=1}^{j} |a_{i_k}| + C_5 e^{-j} \ln^{1/2}(e^{j+1}) \Big(\sum_{k=j+1}^{\infty} |a_{i_k}|^2\Big)^{1/2} \\
&\le e^{-j} \max(D, 2C_5) \left\{ \sum_{k=1}^{j} a_k^* + \sqrt{j} \Big[\sum_{k=j+1}^{\infty} (a_k^*)^2\Big]^{1/2} \right\} \\
&\le e^{-j} \cdot 4 \max(D, 2C_5) \mathcal{K}(\sqrt{j}, a; \ell_1, \ell_2),
\end{aligned}$$

where the last inequality is a corollary of (2.2).

Therefore, relation (8.7) holds for $t = e^{-j}$ $(j \in \mathbb{N})$. Since the functions considered here are concave, it follows that (8.7) holds for all $t \in [0, 1]$. The theorem is proved. □

Remark 8.2. The uniform boundedness of random variables f_i $(i = 1, 2, \ldots)$ in Condition (d) is important. Indeed, consider a system of functions $f_i(t) = \ln^{1/2} e/t \cdot \chi_{[2^{-i}, 2^{-i+1})}(t)$ $(i = 1, 2, \ldots)$ on $[0, 1]$. For arbitrary sequence $a = (a_i)_{i=1}^{\infty} \in \ell_2$, we have

$$\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\|_{L_{N_2}} \le \| \ln^{1/2} e/t \|_{L_{N_2}} \cdot \|a\|_2 = C \|a\|_2,$$

i.e., $\sum_{i=1}^{\infty} a_i f_i \in L_{N_2}$. Nevertheless , $\{f_i\} \overset{\mathbb{P}}{\not\sim} \{r_i\}$.

Our next task is to find conditions under which an arbitrary sequence of random variables is equivalent in distribution to the Rademacher system. It will help us to obtain non-trivial examples of systems $\{f_n\}_{n=1}^{\infty}$ such that $\{f_n\} \overset{\mathbb{P}}{\sim} \{r_n\}$ and to find sufficient and necessary conditions under which, from a sequence of random variables, subsequence equivalent in distribution to the Rademacher system can be extracted.

Theorem 8.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of random variables defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. The following conditions are equivalent to each other*:

(1) $\{f_n\} \overset{\mathbb{P}}{\sim} \{r_n\}$;
(2) *there exists a constant not depending on $t \in [1, \infty)$, $m \in \mathbb{N}$, and $a_n \in \mathbb{R}$ $(n = 1, \ldots, m)$ such that*

$$\Big\|\sum_{n=1}^{m} a_n f_n\Big\|_t \asymp \Big\|\sum_{n=1}^{m} a_n r_n\Big\|_t;$$

(3) *there exists a constant not depending on $t \in [1, \infty)$, $m \in \mathbb{N}$, and $a = (a_n)_{n=1}^{m}$ such that*

$$\Big\|\sum_{n=1}^{m} a_n f_n\Big\|_t \asymp \mathcal{K}(\sqrt{t}, a; \ell_1, \ell_2).$$

Proof. First of all, due to (8.14) the equivalence of (2) ⇔ (3) is obvious and the implication (1) ⇒ (2) immediately follows from the definition of systems equivalent in distribution. Therefore, it remains to prove the implication (3) ⇒ (1).

Using the previous notation, rewrite the condition as follows:

$$B^{-1} \bar{\kappa}_a(t) \le \Big\|\sum_{n=1}^{m} a_n f_n\Big\|_t \le B \bar{\kappa}_a(t), \tag{8.19}$$

where $B > 0$ does not depend on $t \in [1, \infty)$, $m \in \mathbb{N}$, and $a = (a_n)_{n=1}^m$. Note that, due to the previous theorem, we have to show only that $\{r_i\} \overset{\mathbb{P}}{\prec} \{f_i\}$.

For arbitrary $m \in \mathbb{N}$, we introduce

$$f(\omega) = \sum_{n=1}^m a_n f_n(\omega) \ (\omega \in \Omega) \quad \text{and} \quad r(x) = \sum_{n=1}^m a_n r_n(x) \ (x \in [0,1]).$$

Using (8.14) and (8.19) and arguing as in the proof of the implication $(b) \Rightarrow (c)$ in Theorem 8.1, we obtain the relations

$$m\{|r(x)| > z\} \le e^{1-F((C_1')^{-1}z)} \quad \text{if} \quad z > 0, \tag{8.20}$$

$$m\{|r(x)| > z\} = 0 \quad \text{if} \quad z \ge \|a\|_1, \tag{8.21}$$

and

$$\mathbb{P}\{|f(\omega)| > z\} \ge (C_2')^{-1} e^{-F(C_4' z)} \quad \text{if} \quad 0 < z < (16B)^{-1}\|a\|_1, \tag{8.22}$$

where C_1' is a universal constant and C_2' and C_4' depend only on B.

Introduce

$$A' := \ \max(\, C_2' e\, ,\ C_1' C_4'\, ,\ 16B\,). \tag{8.23}$$

Note that the dependence of A' on the system $\{f_n\}_{n=1}^\infty$ is related only to its dependence on B, which is the constant at (8.19).

If $0 < z < (16B)^{-1}\|a\|_1$, then, due to relations (8.20), (8.22), and (8.23) and the increasing of the functional $F(s)$, the following inequality holds:

$$m\{|r(x)| > A'z\} \le e^{1-F((C_1')^{-1}A'z)} \le e^{1-F(C_4' z)} \le C_2' e\, \mathbb{P}\{|f(\omega)| > z\} \le A'\, \mathbb{P}\{|f(\omega)| > z\}. \tag{8.24}$$

If $z \ge (16B)^{-1}\|a\|_1$, then from (8.23), it follows that $A'z \ge \|a\|_1$; therefore, due to (8.21), we have $m\{|r(x)| > A'z\} = 0$. Then inequality (8.24) holds as well. Thus, $\{r_n\} \overset{\mathbb{P}}{\prec} \{f_n\}$. $\square$

Remark 8.3. Note that a similar statement holds for finite sets of random variables as well. Then the proof shows that the constants of the equivalence of conditions (1)–(3) depend only on each other. Thus, e.g., if relation (8.19) holds for sets $\{f_n\}_{n=1}^N$ with a constant not depending on $N = 1, 2, \dots$, then $\{f_n\}_{n=1}^N \overset{\mathbb{P}}{\sim} \{r_n\}_{n=1}^N$ with a constant not depending on $N = 1, 2, \dots$

Apply the obtained results to particular classes of random variables Begin from multiplicative systems (see definitions and examples in Sec. 0.4). Recall that $\mathbb{E}f$ denotes the expectation of the random variable f, i.e., $\mathbb{E}f = \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$.

Theorem 8.3. *Let $\{f_i\}_{i=1}^\infty$ be a multiplicative system of random variables and $\{g_i\}_{i=1}^\infty$ be a strongly multiplicative system of random variables on probability spaces $(\Omega, \Sigma, \mathbb{P})$ and $(\Omega', \Sigma', \mathbb{P}')$ respectively such that for every $i = 1, 2, \dots$ we have $g_i \ne 0$ and*

$$\|f_i\|_\infty \cdot \|g_i\|_\infty \le \mathbb{E}(g_i^2). \tag{8.25}$$

Then for any $1 \le p \le \infty$, $n \in \mathbb{N}$, and real $a_1, a_2, \dots, a_n$ the following inequality holds:

$$\Big\| \sum_{i=1}^n a_i f_i \Big\|_p \le 2 \Big\| \sum_{i=1}^n a_i g_i \Big\|_p. \tag{8.26}$$

Proof. Consider an integral operator

$$Tx(\omega) = \int_{\Omega'} K(\omega, \omega') x(\omega')\, d\mathbb{P}' \ \ (\omega \in \Omega)$$

with a kernel

$$K(\omega, \omega') = \prod_{i=1}^n (1 + \alpha_i f_i(\omega) g_i(\omega')), \quad (\omega, \omega') \in \Omega \times \Omega',$$

where

$$\alpha_i = \frac{1}{\mathbb{E} g_i^2}, \quad i = 1, 2, \ldots$$

We need the properties of $K(\omega, \omega')$. First of all, from (8.25), it follows that

$$K(\omega, \omega') \geq 0 \quad \text{for almost every } (\omega, \omega') \in \Omega \times \Omega'. \tag{8.27}$$

Moreover, due to the multiplicativity of the systems $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$, we have

$$\int\limits_{\Omega'} K(\omega, \omega') \, d\mathbb{P}' = 1 \quad \text{for almost every } \omega \in \Omega \tag{8.28}$$

and

$$\int\limits_{\Omega} K(\omega, \omega') \, d\mathbb{P} = 1 \quad \text{for almost every } \omega' \in \Omega'. \tag{8.29}$$

Note that

$$f_j(\omega) = T g_j(\omega) \tag{8.30}$$

for all $j = 1, 2, \ldots, n$. Indeed, since $\{g_i\}_{i=1}^{\infty}$ is a strongly multiplicative system, taking into account the definition of α_j, we obtain the relation

$$\begin{aligned} T g_j(\omega) &= \int\limits_{\Omega'} \prod_{i=1}^{n} \big(1 + \alpha_i f_i(\omega) g_i(\omega')\big) \cdot g_j(\omega') \, d\mathbb{P}' \\ &= \int\limits_{\Omega'} \Big(g_j(\omega') + \alpha_j f_j(\omega) g_j^2(\omega') + g_j(\omega') \sum_{i \neq j} \alpha_i f_i(\omega) g_i(\omega') \\ &\quad + g_j(\omega') \sum_{k=2}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_k} f_{i_1}(\omega) g_{i_1}(\omega') \ldots f_{i_k}(\omega) g_{i_k}(\omega') \Big) \, d\mathbb{P}' = f_j(\omega). \end{aligned}$$

Due to (8.30), for any real $a_1, a_2, \ldots, a_n$, the following relation holds:

$$\sum_{j=1}^{n} a_j f_j(\omega) = T\Big(\sum_{j=1}^{n} a_j g_j \Big)(\omega). \tag{8.31}$$

Further, relations (8.27) and (8.28) show that T is bounded from $L_\infty(\Omega')$ to $L_\infty(\Omega)$. Indeed, for all $x \in L_\infty(\Omega')$, the following inequality holds:

$$|Tx(\omega)| \leq \int\limits_{\Omega} K(\omega, \omega') |x(\omega')| \, d\mathbb{P}' \leq \int\limits_{\Omega} K(\omega, \omega') \, d\mathbb{P}' \cdot \|x\|_\infty = \|x\|_\infty.$$

Hence, the following equivalent inequality holds:

$$\|T\|_{L_\infty(\Omega') \mapsto L_\infty(\Omega)} \leq 1. \tag{8.32}$$

Now, show that

$$\|T\|_{L_1(\Omega') \mapsto L_1(\Omega)} \leq 1. \tag{8.33}$$

To do that, consider the following operator T' associated to T

$$T' y(t) = \int\limits_{\Omega} K(\omega, \omega') y(\omega) \, d\mathbb{P}, \quad \omega' \in \Omega'.$$

Using (8.27) and (8.29) instead of (8.27) and (8.28), we obtain (in the same way as for T) that

$$\|T'\|_{L_\infty(\Omega) \mapsto L_\infty(\Omega')} \leq 1.$$

Therefore, the positive integral operator T' is bounded from $L_\infty(\Omega)$ to $L_\infty(\Omega')$. Therefore, the operator adjoint to T' coincides with the one associated with T' (see [91, Theorem 2.4.5]); therefore, it coincides with T, which is bounded from $L_1(\Omega')$ to $L_1(\Omega)$. Moreover, we have

$$\|T\|_{L_1(\Omega')\mapsto L_1(\Omega)} = \|T'\|_{L_\infty(\Omega)\mapsto L_\infty(\Omega')};$$

therefore, due to the previous relation, we obtain (8.33).

By the interpolation Riesz–Thorin theorem (see Sec. 0.5 or [40, Theorem 1.1.1] for details), any linear operator A bounded from $L_\infty(\Omega')$ to $L_\infty(\Omega)$ and from $L_1(\Omega')$ to $L_1(\Omega)$ is bounded from $L_p(\Omega')$ to $L_p(\Omega)$ $(1 \le p \le \infty)$ and

$$\|A\|_{L_p(\Omega')\to L_p(\Omega)} \le 2\max\big(\|A\|_{L_\infty(\Omega')\mapsto L_\infty(\Omega)}, \|A\|_{L_1(\Omega')\mapsto L_1(\Omega)}\big).$$

Therefore, inequality (8.26) is a corollary of relations (8.31)–(8.33), and the theorem is proved. □

Theorems 8.1–8.3 imply the following assertion.

Theorem 8.4. *Assume that a multiplicative system of random variables $\{\varphi_n\}_{n=1}^\infty$ defined on a probability space $(\Omega,\Sigma,\mathbb{P})$ is uniformly bounded, i.e.,*

$$|\varphi_n(\omega)| \le D \quad \textit{for all} \ \ n = 1,2,\dots \quad \textit{and} \ \ \omega\in\Omega. \tag{8.34}$$

Then $\{\varphi_n\} \overset{\mathbb{P}}{\prec} \{r_n\}$ with a constant depending only on D.

If $\{\varphi_n\}_{n=1}^\infty$ is strongly multiplicative and

$$d := \inf_{n=1,2,\dots} \mathbb{E}(\varphi_n^2) > 0, \tag{8.35}$$

then $\{\varphi_n\} \overset{\mathbb{P}}{\sim} \{r_n\}$ and the constant of this equivalence depends only on D and d.

Proof. First of all, recall that a Rademacher system is strongly multiplicative (see Corollary 1.2).

If $\{\varphi_n\}$ is multiplicative and uniformly bounded by a constant D, then Condition (8.25) holds for the systems of random variables $f_n = \varphi_n/D$ and $g_n = r_n$ $(n = 1,2,\dots)$. Therefore, applying (8.26), we obtain that

$$\Big\|\sum_{n=1}^m a_n\varphi_n\Big\|_p \le 2D\Big\|\sum_{n=1}^m a_n r_n\Big\|_p$$

for all $p \ge 1$, $m\in\mathbb{N}$, and $a_n\in\mathbb{R}$ $(n = 1,2,\dots,m)$. Thus, Theorem 8.1 and Theorem 2.1 yield the first statement.

Assume that $\{\varphi_n\}_{n=1}^\infty$ is strongly multiplicative and satisfies Condition (8.35). Then (8.25) holds for random variables $f_n = (d/D)r_n$ and $g_n = \varphi_n$. Therefore, from Theorem 8.3, we obtain the inequality inverse to the previous one, i.e.,

$$\Big\|\sum_{n=1}^m a_n r_n\Big\|_p \le \frac{2D}{d}\Big\|\sum_{n=1}^m a_n\varphi_n\Big\|_p.$$

To complete the proof, we apply Theorem 8.2. □

From the last theorem and the definition of the symmetric space, we get the following assertion.

Corollary 8.2. *Let $\{\varphi_n\}_{n=1}^\infty$ be a multiplicative system of random variables on a probability space $(\Omega,\Sigma,\mathbb{P})$, satisfying Condition* (8.34). *Then for any symmetric space X on $[0,1]$, all $n\in\mathbb{N}$, and arbitrary real $a_1,a_2,\dots,a_n$, the following inequality holds*:

$$\Big\|\sum_{i=1}^n a_i\varphi_i\Big\|_X \le C\Big\|\sum_{i=1}^n a_i r_i\Big\|_X,$$

where $C > 0$ depends only on D.

If $\{\varphi_n\}_{n=1}^\infty$ is strongly multiplicative and (8.35) *holds, then for any symmetric space X on $[0,1]$, all $n \in \mathbb{N}$, and arbitrary real $a_1, a_2, \dots, a_n$, we have*

$$\Big\| \sum_{i=1}^n a_i \varphi_i \Big\|_X \asymp \Big\| \sum_{i=1}^n a_i r_i \Big\|_X$$

and the constant of the equivalence depends only on D and d.

In the particular case of independent functions (see Example 0.2), we obtain the following statement.

Corollary 8.3. *Any sequence of independent random variables $\{f_n\}_{n=1}^\infty$, $\mathbb{E}(f_n) = 0$, satisfying Conditions* (8.34) *and* (8.35) *is equivalent in distribution to the Rademacher system.*

Consider another case now. Let G be a compact Abelian group with probability Haar measure μ, and Γ be a group of characters defined on G, where $C(G)$ is the space of continuous complex functions on G. According to the definition given in Sec. 0.4, the set $F \subset \Gamma$ is called a *Sidon set* if there exists $C_F > 0$ such that

$$\sum_{\gamma \in \Gamma} |\widehat{f}(\gamma)| \le C_F \, \|f\|_\infty$$

for any $f \in C(G)$ such that

$$\widehat{f}(\gamma) := \int_G f(s)\gamma(-s)\, d\mu \;=\; 0, \quad \text{if} \quad \gamma \notin F.$$

Prove the following statement similar (in a way) to Theorem 8.3.

Theorem 8.5. *Let $\mathcal{S}$ and $\mathcal{T}$ be compact Abelian groups with probability Haar measures μ and ν respectively. Assume that $(f_j)_{j=1}^\infty$ is a sequence of characters on $\mathcal{S}$ and $(g_j)_{j=1}^\infty$ is a sequence of characters on $\mathcal{T}$ such that there exist constants $C > 0$ and $c > 0$ such that*

$$c \cdot \sup_{t \in \mathcal{T}} \Big| \sum_{i=1}^n a_i f_i(t) \Big| \le \sup_{s \in \mathcal{S}} \Big| \sum_{i=1}^n a_i g_i(s) \Big| \le C \cdot \sup_{t \in \mathcal{T}} \Big| \sum_{i=1}^n a_i f_i(t) \Big| \tag{8.36}$$

for any $n \in \mathbb{N}$ and complex numbers $a_1, a_2, \dots, a_n$.

Then for any $1 \le p \le \infty$, $n \in \mathbb{N}$, and complex numbers $a_1, a_2, \dots, a_n$, the following inequality holds:

$$\frac{c}{C} \cdot \Big\| \sum_{i=1}^n a_i f_i \Big\|_p \le \Big\| \sum_{i=1}^n a_i g_i \Big\|_p \le \frac{C}{c} \cdot \Big\| \sum_{i=1}^n a_i f_i \Big\|_p. \tag{8.37}$$

Proof. First of all, recall that, due to relation (8.36), any mapping U such that $Uf_j = g_j$ $(j = 1, 2, \dots)$ is extended to an isomorphism of the closed linear span $[f_j]$ considered with the norm of $C(\mathcal{S})$ onto the closed linear span $[g_j]$ with the norm of $C(\mathcal{T})$; thus the inequalities $\|U\| \le C$ and $\|U^{-1}\| \le c^{-1}$ are satisfied.

For a fixed $t \in \mathcal{T}$, by δ_t denote the Dirac measure generated by the point t, and by φ_t^* denote the restriction of δ_t to $[g_j]$. Obviously, φ_t^* is a linear bounded functional on $[g_j]$ and $\|\varphi_t^*\| \le 1$. Moreover, for every $j = 1, 2, \dots$, we have

$$g_j(t) = \varphi_t^*(g_j) = \varphi_t^*(Uf_j) = (U^*\varphi_t^*)(f_j). \tag{8.38}$$

Let μ_t be the complex Borel measure on $\mathcal{S}$. By the Riesz theorem, it generates an extension of the linear functional $U^*\varphi_t^*$ to the space $C(\mathcal{S})$. Such an extension exists due the Hahn–Banach theorem. Define measures μ_t' and $\overline{\mu_t}$ on Borel sets $V \subset \mathcal{S}$:

$$\mu_t'(V) = \mu_t(-V); \quad \overline{\mu}_t(V) = \overline{\mu_t(V)}.$$

Note that the full measure variations μ_t' and $\overline{\mu}_t$ satisfy the estimate

$$\|\overline{\mu}_t\| = \|\mu_t'\| = \|\mu_t\| \le \|U^*\| = \|U\| \le C. \tag{8.39}$$

Moreover, due to (8.38), the Fourier coefficients of these measures corresponding to f_j $(j = 1, 2, \dots)$ are defined by the relations

$$\widehat{\mu_t'}(f_j) = \int\limits_{\mathcal{S}} f_j(-s)\, d\mu_t' = \int\limits_{\mathcal{S}} f_j(s)\, d\mu_t = g_j(t) \tag{8.40}$$

and

$$\widehat{\overline{\mu}}_t(f_j) = \int\limits_{\mathcal{S}} f_j(-s)\, d\overline{\mu}_t = \int\limits_{\mathcal{S}} \overline{f_j(s)}\, d\overline{\mu_t} = \overline{g_j(t)} = g_j(-t). \tag{8.41}$$

For arbitrary complex $a_1, a_2, \dots, a_n$ and $t \in \mathcal{T}$, we define

$$f(s) = \sum_{i=1}^{n} a_i f_i(s) \quad \text{and} \quad f_t(s) = \sum_{i=1}^{n} a_i g_i(t) f_i(s) \quad (s \in \mathcal{S}).$$

Then, if $f * \mu$ is the convolution of the measure μ with a function f, i.e.,

$$(f * \mu)(s) = \int\limits_{\mathcal{S}} f(s - \sigma)\, d\mu(\sigma) \quad (s \in \mathcal{S}),$$

then a comparison of the Fourier coefficients and relations (8.40) and (8.41) yields the following relations:

$$f * \mu_t' = f_t \quad \text{and} \quad f_t * \overline{\mu}_t = f.$$

Therefore, due to the well-known Young inequality and relations (8.39), the inequalities

$$\|f_t\|_p \le \|\mu_t'\| \|f\|_p \le \|U\|\, \|f\|_p \le C\|f\|_p \quad \text{and} \quad \|f\|_p \le \|\overline{\mu}_t\| \|f_t\|_p \le \|U\|\, \|f_t\|_p \le C\|f_t\|_p$$

hold, where $\|f\|_p = (\int\limits_{\mathcal{S}} |f|^p\, d\mu)^{1/p}$. Integrating the last inequalities in the measure ν on $\mathcal{T}$ and applying the Fubini theorem, we obtain the inequality

$$C^{-1}\|f\|_p \le \Big(\int\limits_{\mathcal{S}} \int\limits_{\mathcal{T}} \Big| \sum_{j=1}^{n} a_j f_j(s) g_j(t) \Big|^p d\nu\, d\mu \Big)^{1/p} \le C\|f\|_p.$$

Changing the places of $\{f_j\}$ and $\{g_j\}$, we obtain that for $g(t) = \sum\limits_{j=1}^{n} a_j g_j(t)$, the following relation holds:

$$c\|g\|_p \le \Big(\int\limits_{\mathcal{S}} \int\limits_{\mathcal{T}} \Big| \sum_{j=1}^{n} a_j f_j(s) g_j(t) \Big|^p d\nu\, d\mu \Big)^{1/p} \le c^{-1}\|g\|_p.$$

Since (8.37) is a corollary of the two last inequalities, the theorem is proved. □

it is well known (see comments to Chap. 1) that the Rademacher system (more exactly, any system equivalent in distribution) can be considered as a sequence of characters defined on the compact Abelian Cantor group $\mathcal{K} := \{-1, 1\}^{\mathbb{N}}$. Moreover, one can easily check (see Remark 1.5) that for arbitrary $n \in \mathbb{N}$ and complex $a_1, a_2, \dots, a_n$, the relation

$$\frac{1}{2} \sum_{i=1}^{n} |a_i| \le \sup_{0 \le s \le 1} \Big| \sum_{i=1}^{n} a_i r_i(s) \Big| \le \sum_{i=1}^{n} |a_i|$$

holds. Therefore, we obtain the following assertion.

Corollary 8.4. *If $F = \{\gamma_n\} \subset \Gamma$ is a Sidon set, then there exists a constant depending only on the Sidon constant C_F such that*

$$\Big\|\sum_{n=1}^{m} a_n\gamma_n\Big\|_p \asymp \Big\|\sum_{n=1}^{m} a_n r_n\Big\|_p \quad (p \geq 1).$$

The following result is a corollary of the last statement and Theorem 8.2.

Theorem 8.6. *Any infinite Sidon system of characters $F = \{\gamma_n\}_{n=1}^{\infty}$ defined on a compact Abelian group is equivalent in distribution to the Rademacher system, and the constant of the equivalence depends only on the Sidon constant C_F.*

Since the sequence $\{e^{ik_nt}\}_{n=1}^{\infty}$ is a Sidon system of characters on the torus $\mathbb{T}$ provided that $k_{n+1}/k_n \geq \lambda > 1$ (see [130]), it follows that the last result holds both for the specified system and for its real case, i.e., the lacunary trigonometric system (see Example 0.1).

Corollary 8.5. *The sequences $f_n(x) = \sin(2\pi k_n x)$ and $g_n(x) = \cos(2\pi k_n x)$, $0 \leq x \leq 1$ are equivalent in distribution to the Rademacher system provided that $k_{n+1}/k_n \geq \lambda > 1$.*

Comments and References. For the notion of majorization (domination) in distribution and various comparison results of sums of independent random variables, see [137, Sec. 5.4] and [94, Chapter 3]. Probably, the most important question in this area is as follows: when from distributional estimates for summands follow a similar estimate for their sum? We provide a well-known and important result (the Kwapien–Rychlic theorem, see [137, Theorem 5.4.4]). Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be sequences of independent symmetric random variables, and let there exist positive constants C and α such that

$$\mathbb{P}\{|g_i(\omega)| > r\} \leq C\mathbb{P}\{\alpha|f_i(\omega)| > r\}$$

for all $r > 0$ and $i \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, arbitrary real $x_1, x_2, \dots, x_n$, and $r > 0$, the following inequality holds:

$$\mathbb{P}\Big\{\Big|\sum_{i=1}^{n} x_i g_i(\omega)\Big| > r\Big\} \leq 2C\mathbb{P}\Big\{C\alpha\Big|\sum_{i=1}^{n} x_i f_i(\omega)\Big| > r\Big\}.$$

Note that the last statement holds even if $x_1, x_2, \dots, x_n$ belong to an arbitrary Banach space (obviously, absolute values are substituted for norms in the last inequality). Proposition 8.1 and Theorem 8.1 (without (b)) are proved in [15]. Theorem 8.2 and an idea of the reduction of the equivalence in the distribution problem of an arbitrary system and the Rademacher system to the comparison of L_p-norms of polynomials with respect to these systems is contained [18]). Theorem 8.3 belongs to Jakubowski and Kwapien (see [77]). We give another (in a way, more direct) proof. Theorem 8.4 and Corollaries 8.2 and 8.3 are taken from [18]. Theorem 8.5 is proved in [116]. Corollary 8.4 is proved in a different way in [118, Théorème 2.1]. Theorem 8.6 is proved in a different way in [4, Theorem 1.3]. It is a corollary of the Pchelinskiy theorem and Theorem 8.2. The last statement of this chapter is Corollary 8.5. It can be considered to be a result of rather long previous investigations. An analogy between the behavior of Rademacher series and lacunary trigonometric series is a basis to find a deeper connection between them. It leads to the study of systems equivalent in distribution to the "model" Rademacher system. At the end of the 70s to the beginning of the 80s, it became clear that norms of polynomials with respect to the Rademacher system and lacunary trigonometric polynomials are equivalent in L_p-spaces. Many works were devoted to extending these results to various classes of symmetric spaces. A good example is a remark of Belov and Rodin (see [38]): Using Theorem 8.3, the authors extend the results of [124] about Rademacher series to lacunary trigonometric series. Corollary 8.5 shows that the equivalence of norms of polynomials on the Rademacher system and lacunary trigonometric polynomials takes place for all symmetric spaces It is a corollary of a more fundamental fact: the equivalence of these systems of functions in distribution.

Chapter 9

SELECTION OF LACUNARY SUBSYSTEMS

The main aim of this chapter is to find conditions ensuring that a sequence of random variables defined on an arbitrary probability space contains a subspace equivalent in distribution to the Rademacher system considered on a segment $[0, 1]$. We will see that for uniformly bounded systems, they actually coincide with the conditions of classical theorems about extracting of Sidon subsequences and S_∞-subsystems (see Sec. 0.4 and comments at the end of this chapter). Therefore, since any sequence equivalent in distribution to a Rademacher system has the same properties, it follows that the obtained results lead to classical results. The equivalence in distribution to the Rademacher system is (in a sense) the best possible lacunarity which can be obtained by a thinning of a sequence. In this chapter, we are also interested in the quantitative aspect of this phenomenon, i.e., the density of lacunary subsystems that can be extracted from finite sets of measurable functions.

The following assertion is one of the most important in this chapter.

Theorem 9.1. *Let a system of random variables $\{f_n\}_{n=1}^\infty$ be defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. A subsystem $\{\varphi_i\}_{i=1}^\infty$ equivalent in distribution to the Rademacher system on $[0, 1]$ can be extracted from $\{f_n\}$ if and only if there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ satisfying the following conditions*:

(1) $|f_{n_k}(\omega)| \le D$ $(k = 1, 2, \dots;\ \omega \in \Omega)$;
(2) $f_{n_k} \to 0$ *weakly in* L_2;
(3) $d = \inf_{k=1,2,\dots} \|f_{n_k}\|_2 > 0$.

The equivalence constant depends only on D and d.

A "hard" part of the proof is the proof of the sufficiency of Conditions (1)–(3) for the existence of a subsequences $\{\varphi_k\} \overset{\mathbb{P}}{\sim} \{r_k\}$. We do it in two steps: first, we show that the assumptions of the theorem allow us to extract a subsequence $\{\varphi_k\} \subset \{f_n\}$ majorized by distribution by the Rademacher system; then we prove the inverse statement, i.e., prove that there exists a subsystem majorizing it. Begin from an auxiliary statement, denoting by $\mathbb{E}[f\,|\,g_1, \dots, g_n]$ the conditional expectation of random variable f with respect to the σ-subalgebra of the σ-algebra Σ generated by the random variables $g_1, g_2, \dots, g_n$ (see Sec. 0.3).

Lemma 9.1. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of random variables defined on a probability space $(\Omega, \Sigma, \mathbb{P})$, $|f_n(\omega)| \le D$ $(\omega \in \Omega, n \in \mathbb{N})$, and $f_n \to 0$ weakly in $L_2(\Omega)$. Then there exist sequences $\{f_{n_k}\} \subset \{f_n\}_{n=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ such that*

(i) $|g_k(\omega)| \le 2D$, $\omega \in \Omega, k \in \mathbb{N}$;
(ii) $\mathbb{E}[g_k|\,g_1, \dots, g_{k-1}] = 0$ *a.e. on* Ω;
(iii) $\|f_{n_k} - g_k\|_k \le 2^{-k}$, $k \in \mathbb{N}$.

Proof. We use induction. Since $f_n \to 0$ weakly in $L_2(\Omega)$, it follows that there exists $n_1 \in \mathbb{N}$ such that

$$|\mathbb{E}[f_{n_1}]| \le \frac{1}{6}. \tag{9.1}$$

Then there exist a finite-valued function h_1, $|h_1(\omega)| \le D$ $(\omega \in \Omega)$, such that

$$\mathbb{E}[|f_{n_1} - h_1|] \le \frac{1}{6}. \tag{9.2}$$

Now, if $g_1 := h_1 - \mathbb{E}[h_1]$, then $\mathbb{E}[g_1] = 0$ and $|g_1(\omega)| \le 2D$. Moreover, from inequalities (9.1) and (9.2), it follows that

$$\begin{aligned}\|f_{n_1} - g_1\|_1 &\le \mathbb{E}[|f_{n_1} - h_1|] + \mathbb{E}[|h_1 - g_1|] = \mathbb{E}[|f_{n_1} - h_1|] + |\mathbb{E}[h_1]| \\ &\le 2\mathbb{E}[|f_{n_1} - h_1|] + |\mathbb{E}[f_{n_1}]| \le \frac{1}{2}.\end{aligned}$$

Now, assume that $k > 1$ and $f_{n_1}, \dots, f_{n_{k-1}}$, $1 \le n_1 < \dots < n_{k-1}$, and $g_1, \dots, g_{k-1}$ are such that for all $s = 1, 2, \dots, k-1$ the following conditions are satisfied:

(a) $|g_s(\omega)| \le 2D$;

(b) any g_s is constant on sets $e_i^{(s)} \subset \Omega$, $i = 1, \dots, i_s$, such that $e_i^{(s)} \bigcap e_j^{(s)} = \varnothing\ (i \ne j)$, $\bigcup_i e_i^{(s)} = \Omega$, and any $e_i^{(s)}$ is a subset of the constancy set $e_i^{(s-1)}$, $i = 1, \dots, i_{s-1}$, for the function g_{s-1};

(c) $\mathbb{E}[g_s | g_1, \dots, g_{s-1}] = 0$ a.e. on Ω;

(d) $\|f_{n_s} - g_s\|_s \le 2^{-s}$.

Find the next couple f_{n_k} and g_k satisfying similar conditions. Since $f_n \to 0$ weakly in $L_2(\Omega)$, it follows that there exists $n_k > n_{k-1}$ such that for all $i = 1, \dots, i_{k-1}$ the following inequality holds:

$$\Big| \int\limits_{e_i^{(k-1)}} f_{n_k}(\omega)\, d\mathbb{P}(\omega) \Big| \le \frac{2^{-k}}{3}\, \mathbb{P}(e_i^{(k-1)}). \tag{9.3}$$

Let h_k be a finite-valued function, $|h_k(\omega)| \le D$, all constancy sets of this function are contained in the sets $e_i^{(k-1)}$ $(i = 1, \dots, i_{k-1})$, and the following inequality holds for all $i = 1, \dots, i_{k-1}$:

$$\int\limits_{e_i^{(k-1)}} |f_{n_k}(\omega) - h_k(\omega)|^k\, d\mathbb{P}(\omega) \le \left(\frac{2^{-k}}{3}\right)^k \mathbb{P}(e_i^{(k-1)}). \tag{9.4}$$

Assume that $g_k = h_k - \mathbb{E}[h_k |\, g_1, \dots, g_{k-1}]$. Then $\mathbb{E}[g_k |\, g_1, \dots, g_{k-1}] = 0$ a.e. on Ω and $|g_k(\omega)| \le 2D$. Since for $\omega \in e_i^{(k-1)}$ and arbitrary random variable f on Ω, we have

$$\mathbb{E}[f |\, g_1, \dots, g_{k-1}](\omega) = \mathbb{P}(e_i^{(k-1)})^{-1} \int\limits_{e_i^{(k-1)}} f(u)\, d\mathbb{P}(u),$$

it follows from (9.4) that

$$|\mathbb{E}[(f_{n_k} - h_k)|\, g_1, \dots, g_{k-1}]| \le \max_{i=1,\dots,i_{k-1}} \mathbb{P}(e_i^{(k-1)})^{-1} \int\limits_{e_i^{(k-1)}} |f_{n_k}(u) - h_k(u)|\, d\mathbb{P}(u)$$

$$\le \max_{i=1,\dots,i_{k-1}} \mathbb{P}(e_i^{(k-1)})^{-1/k} \Big(\int\limits_{e_i^{(k-1)}} |f_{n_k}(u) - h_k(u)|^k\, d\mathbb{P}(u) \Big)^{1/k} \le \frac{2^{-k}}{3}.$$

Therefore, using (9.3), we obtain the inequality

$$|\mathbb{E}[h_k |\, g_1, \dots, g_{k-1}]| \le |\mathbb{E}[(f_{n_k} - h_k)|\, g_1, \dots, g_{k-1}]| + |\mathbb{E}[f_{n_k} |\, g_1, \dots, g_{k-1}]| \le \frac{2^{-k+1}}{3}.$$

Then, from (9.4), it follows that

$$\|f_{n_k} - g_k\|_k \le \|f_{n_k} - h_k\|_k + \|E[h_k |\, g_1, \dots, g_{k-1}]\|_\infty \le \frac{2^{-k}}{3} + \frac{2^{-k+1}}{3} = 2^{-k}.$$

The proof is completed. ☐

In the sequel, $\kappa_a(t) := \mathcal{K}(t, a; \ell_1, \ell_2)$, where $a \in \ell_2$ and $t > 0$.

Theorem 9.2. *Assume that $\{f_n\}_{n=1}^\infty$ is a sequence of random variables defined on a probability space $(\Omega, \Sigma, \mathbb{P})$, $|f_n(\omega)| \le D$ $(\omega \in \Omega, n \in N)$, and $f_n \to 0$ weakly in $L_2(\Omega)$. Then a subsequence $\{f_{n_k}\}$ can be extracted from $\{f_n\}_{n=1}^\infty$ such that*

$$\Big\| \sum_{i=1}^\infty a_i f_{n_i} \Big\|_t \le C \kappa_a(\sqrt{t}) \qquad (t \ge 1).$$

Proof. By Lemma 9.1, there exist subsequences $\{f_{n_k}\} \subset \{f_n\}$ and $\{g_k\}$ with the following properties: $|g_k(\omega)| \le 2D$, $\mathbb{E}[g_k|\, g_1, \ldots, g_{k-1}] = 0$ a.e., and

$$\|g_k - f_{n_k}\|_k \le 2^{-k} \quad (k \in \mathbb{N}). \tag{9.5}$$

Note that $\{g_k\}$ is multiplicative. Indeed, due to the properties of the conditional expectation (see [65, Lemmas 1.11 and 1.13]), for arbitrary $k_1 < \ldots < k_m$, the following relation holds:

$$\begin{aligned}\mathbb{E}[g_{k_1} \ldots g_{k_{m-1}} g_{k_m} |\, g_{k_1}, \ldots, g_{k_{m-1}}] &= g_{k_1} \ldots g_{k_{m-1}} \mathbb{E}[g_{k_m} |\, g_{k_1}, \ldots, g_{k_{m-1}}] \\ = g_{k_1} \ldots g_{k_{m-1}} \mathbb{E}\left[\mathbb{E}[g_{k_m} |\, g_1 g_2 \ldots, g_{k_m} - 1] \,|\, g_{k_1}, \ldots, g_{k_{m-1}}\right] &= 0;\end{aligned}$$

therefore, we have

$$\mathbb{E}[g_{k_1} \ldots g_{k_{m-1}} g_{k_m}] = \int_\Omega \mathbb{E}[g_{k_1} \ldots g_{k_{m-1}} g_{k_m} |\, g_{k_1} \ldots g_{k_{m-1}}]\, d\mathbb{P}(\omega) \; = \; 0.$$

Therefore, by Theorem 8.3, the following inequality holds for arbitrary $(b_k) \in \ell_2$ and $p \ge 1$:

$$\Big\| \sum_{k=1}^{\infty} b_k g_k \Big\|_p \le 2D \Big\| \sum_{k=1}^{\infty} b_k r_k \Big\|_p. \tag{9.6}$$

Therefore, applying relation (9.5), Theorem 1.3, and inequality (2.2), for any $t \in \mathbb{N}$ we obtain the following inequality:

$$\begin{aligned}\Big\| \sum_{i=1}^{\infty} a_i f_{n_i} \Big\|_t &\le \Big\| \sum_{i=1}^{t} a_i f_{n_i} \Big\|_t + \Big\| \sum_{i=t+1}^{\infty} a_i (f_{n_i} - g_i) \Big\|_t + \Big\| \sum_{i=t+1}^{\infty} a_i g_i \Big\|_t \\ &\le D \sum_{i=1}^{t} |a_i| + \Big(\sum_{i=t+1}^{\infty} a_i^2 \Big)^{1/2} \sum_{i=t+1}^{\infty} \|f_{n_i} - g_i\|_i + 2D \Big\| \sum_{i=t+1}^{\infty} a_i r_i \Big\|_t \\ &\le D \sum_{i=1}^{t} |a_i| + (2D + 2^{-t}) \sqrt{t} \Big(\sum_{i=t+1}^{\infty} a_i^2 \Big)^{1/2} \le 4(2D+1)\kappa_a(\sqrt{t}).\end{aligned}$$

Since the last inequality is exactly extended for the case of any real $t \ge 1$, it follows that the theorem is proved. □

The next three results follow from Theorem 9.2 and Theorem 8.1. In the last one, we use the weak relative compactness of bounded sets in L_2 as well.

Corollary 9.1. *If $\{f_n\}_{n=1}^\infty$ is a sequence of random variables satisfying the assumptions of Theorem 9.2, then there exists its subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\} \overset{\mathbb{P}}{\prec} \{r_k\}$ with a constant depending only on D.*

Corollary 9.2. *If $\{f_n\}_{n=1}^\infty$ is a sequence of random variables satisfying the conditions of Theorem 9.2, then there exists its subsequence $\{f_{n_k}\}$ such that $\exp(\lambda \sum_{k=1}^{\infty} a_k r_{n_k})^2 \in L_1$ for any $(a_k)_{k=1}^\infty \in \ell_2$ and $\lambda \in \mathbb{R}$.*

Corollary 9.3. *If $\{f_n\}_{n=1}^\infty$ is a uniformly bounded orthonormal sequence of random variables defined on a probability space $(\Omega, \Sigma, \mathbb{P})$, then there exists a subsequence $\{\varphi_i\}_{i=1}^\infty$ majorized in distribution by the Rademacher system.*

The following statement is important for the proof of Theorem 9.1.

Theorem 9.3. *Let a system $\{f_n\}_{n=1}^{\infty}$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ satisfy Conditions* (1)–(3) *as a subsequence $\{f_{n_k}\}$ in Theorem* 9.1. *Then there exists a subsequence $\{\varphi_i\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ and a constant $C > 0$ depending only on D and d such that for all $t \geq 1$ and $a = (a_i)_{i=1}^{\infty} \in \ell_2$ the following inequality holds:*

$$\kappa_a(\sqrt{t}) \leq C \left\| \sum_{i=1}^{\infty} a_i \varphi_i \right\|_t. \tag{9.7}$$

Proof. Use the following approximation of the $\mathcal{K}$-functional $\kappa_a(t)$ (see Chap. 2): for arbitrary $a = (a_n)_{n=1}^{\infty} \in \ell_2$ and $t^2 \in \mathbb{N}$, the following inequality holds:

$$\|a\|_{Q(t^2)} \leq \kappa_a(t) \leq \sqrt{2}\, \|a\|_{Q(t^2)}, \tag{9.8}$$

where

$$\|a\|_{Q(t)} = \sup \Big\{ \sum_{j=1}^{t} \Big(\sum_{n \in A_j} a_n^2 \Big)^{1/2} \Big\} \;\; (a = (a_n)_{n=1}^{\infty} \in \ell_2, t \in \mathbb{N}) \tag{9.9}$$

(the supremum is taken over all partitions $\{A_j\}_{j=1}^{t}$ of the set of positive integers $\mathbb{N}$). Also, we will need an estimate from above for L_q-norms ($q > 1$) of modified Riesz products.

First of all, $0 < d \leq \|f_n\|_2 \leq D$ ($n = 1, 2, \ldots$) by assumption. Thus, due to the positive homogeneity of both sides of relation (9.7) with respect to $a \in \ell_2$, we can assume that $\|f_n\|_2 = 1$ ($n = 1, 2, \ldots$); therefore, $D \geq 1$.

Let a sequence of reals $(\varepsilon_i)_{i=1}^{\infty}$ be such that

$$0 < \varepsilon_i < \frac{1}{16} \quad \text{and} \quad \sum_{k=i+1}^{\infty} \varepsilon_k < \varepsilon_i \quad (i = 1, 2, \ldots). \tag{9.10}$$

Since the sequence f_n^2 is weakly relatively compact in L_2 by assumption, it follows that there exists $\{h_k\} \subset \{f_n\}$ such that $h_k^2 \to h$ weakly in this space. Then, due to the properties $\{h_k\}$, we have $0 \leq h(\omega) \leq D^2$ and $\mathbb{E}(h) = 1$. Moreover, $h_k \to 0$ weakly in L_2; therefore, we can find a number k_1 such that for the function $\varphi_1 = h_{k_1}$, the following inequality holds:

$$|\mathbb{E}(\varphi_1)| + |\mathbb{E}(h\varphi_1)| + |\mathbb{E}(\varphi_1^2 - h)| \leq \frac{\varepsilon_1}{2D}.$$

Assume that numbers $k_1 < k_2 < \ldots < k_{i-1}$ and functions $\varphi_1 = h_{k_1}, \varphi_2 = h_{k_2}, \ldots, \varphi_{i-1} = h_{k_{i-1}}$ ($i \geq 2$) are already chosen. Find $k_i > k_{i-1}$ such that for $\varphi_i = h_{k_i}$ the following relation holds:

$$\begin{aligned} \sum \Big\{ &|\mathbb{E}(\varphi_{j_1} \ldots \varphi_{j_s} \varphi_i)| + |\mathbb{E}(h\varphi_{j_1} \ldots \varphi_{j_s} \varphi_i)| + |\mathbb{E}[\varphi_{j_1} \ldots \varphi_{j_s} (\varphi_i^2 - h)]| \\ &+ \sum_{l=1}^{s} |\mathbb{E}(\varphi_{j_1} \ldots \varphi_{j_{l-1}} \varphi_{j_l}^2 \varphi_{j_{l+1}} \ldots \varphi_{j_s} \varphi_i)| \Big\} \leq \frac{\varepsilon_i}{2D}, \end{aligned} \tag{9.11}$$

where $\varphi_{j_0} = \varphi_{j_{s+1}} = 1$ and the sum is taken over all sets of indices

$$1 \leq j_1 < j_2 < \cdots < j_l < \cdots < j_s \leq i - 1 \;\; (s = 1, 2, \ldots, i-1).$$

Note that, due to Theorem 9.2, we can assume that any function

$$\varphi(\omega) = \sum_{i=1}^{\infty} a_i \varphi_i(\omega), \; a = (a_i)_{i=1}^{\infty} \in \ell_2,$$

belongs to L_p for any $1 \leq p < \infty$. Show that the sequence $\{\varphi_i\}_{i=1}^{\infty}$ extracted as above satisfies (9.7).

First, let $t \in \mathbb{N}$ and $\{A_j\}_{j=1}^{t}$ be an arbitrary partition of the set $\mathbb{N}$. For $N \in \mathbb{N}$ and $j = 1, 2, \ldots, t$, define sets $A_j^N = \{i = 1, \ldots, N : i \in A_j\}$ (some of them may be empty). Define Riesz products consisting of blocks corresponding to sets A_j^N:

$$R_N(\omega) = \prod_{i=1}^{N}(1+b_i\varphi_i(\omega)) = \prod_{j=1}^{t}\prod_{i\in A_j^N}(1+b_i\varphi_i(\omega)),$$

where $b_i \in \mathbb{R}$ satisfies the condition

$$\sum_{i\in A_j} b_i^2 \le D^{-2} \quad (j=1,2,\dots,t). \tag{9.12}$$

Following the idea of the classical Sidon method, estimate the following integral from below:

$$I_N = \int\limits_{\Omega} R_N(\omega)\varphi(\omega)\,d\mathbb{P}(\omega) = \sum_{i=1}^{\infty} a_i\mathbb{E}(R_N\varphi_i) = \sum_{i=1}^{\infty} a_i\beta_{i,N}, \tag{9.13}$$

where

$$\begin{aligned}\beta_{i,N} = \mathbb{E}(R_N\varphi_i) = \int\limits_{\Omega}\Big[1+\sum_{k=1}^{N} b_k\varphi_k(\omega) \;+ \sum_{1\le k_1<k_2\le N} b_{k_1}b_{k_2}\varphi_{k_1}(\omega)\varphi_{k_2}(\omega)\,d\mathbb{P}(\omega)\\ \cdots + b_1b_2\dots b_N\varphi_1(\omega)\dots\varphi_N(\omega)\Big]\varphi_i(\omega)\,d\mathbb{P}(\omega) \;=\; b_i\alpha_{i,N}+\gamma_{i,N},\end{aligned} \tag{9.14}$$

$\alpha_{i,N}=1$ $(i\le N)$, $\alpha_{i,N}=0$ $(i>N)$, and

$$\begin{aligned}\gamma_{i,N} = \mathbb{E}(\varphi_i) + \sum_{k=1,k\ne i}^{N} b_k\mathbb{E}(\varphi_k\varphi_i) + \sum_{1\le k_1<k_2\le N} b_{k_1}b_{k_2}\mathbb{E}(\varphi_{k_1}\varphi_{k_2}\varphi_i)\\ \dots + \sum_{1\le k_1<\dots<k_s\le N} b_{k_1}\dots b_{k_s}\mathbb{E}(\varphi_{k_1}\dots\varphi_{k_s}\varphi_i) + \dots + b_1\dots b_N\mathbb{E}(\varphi_1\dots\varphi_N\varphi_i).\end{aligned}$$

Then

$$\gamma_{i,N} = S_1^i + S_2^i + S_3^i, \tag{9.15}$$

where the sums S_k^i $(k=1,2,3)$ are defined as follows. For $k_1<k_2<\dots<k_s<i$, we assume that

$$\mathbb{E}(\varphi_{k_1}\dots\varphi_{k_s}\varphi_i^2) = \mathbb{E}(\varphi_{k_1}\dots\varphi_{k_s}h) + \mathbb{E}[\varphi_{k_1}\dots\varphi_{k_s}(\varphi_i^2-h)].$$

Then we include in the sum S_1^i all summands containing integrals of the form $\mathbb{E}(\varphi_{k_1}\dots\varphi_{k_s}\varphi_i)$ or $\mathbb{E}[\varphi_{k_1}\dots$
$\dots\varphi_{k_s}(\varphi_i^2-h)]$, where $k_1<k_2<\dots<k_s<i$. Include in S_2^i all summands with integrals $\mathbb{E}(\varphi_{k_1}\dots$ $\dots\varphi_{k_{l-1}}\varphi_i\varphi_{k_l}\dots\varphi_{k_s})$, where $k_1<k_2<\dots<k_{l-1}\le i<k_l<\dots<k_s$, $1\le l\le s$. Include in S_3^i all summands with integrals $\mathbb{E}(\varphi_{k_1}\dots\varphi_{k_s}h)$.

Due to (9.12), we have $|b_i|\le 1$ $(i=1,2,\dots)$. Therefore, from (9.11) and (9.10), it follows that

$$|S_1^i| \le \frac{\varepsilon_i}{D} \quad\text{and}\quad |S_2^i| \le \frac{1}{D}\sum_{k=i+1}^{\infty}\varepsilon_k \;<\; \frac{\varepsilon_i}{D}. \tag{9.16}$$

Every summand of S_3^i contains a factor b_i. Therefore, using (9.11) and (9.10) again, we obtain the inequality

$$|S_3^i| \le \frac{|b_i|}{D}\sum_{i=1}^{\infty}\varepsilon_i \le \frac{|b_i|}{8}. \tag{9.17}$$

Note that if $i>N$, then the right-hand side of (9.15) does not contain S_2^i and S_3^i, whence

$$\sum_{i=N+1}^{\infty}|\gamma_{i,N}| \le \frac{1}{D}\sum_{i=N+1}^{\infty}\varepsilon_i \le \frac{1}{16D}. \tag{9.18}$$

Now, let $1 \le i \le N$. For every $j = 1, 2, \ldots, t$, due to relations (9.15)–(9.17), (9.10), and (9.12), the following inequality holds:

$$\Big(\sum_{i \in A_j^N} \gamma_{i,N}^2\Big)^{1/2} \le \sum_{k=1}^{3}\Big(\sum_{i \in A_j^N} (S_k^i)^2\Big)^{1/2} \le \frac{2}{D}\sum_{i \in A_j^N} \varepsilon_i + \frac{1}{8}\Big(\sum_{i \in A_j^N} b_i^2\Big)^{1/2} \le \frac{3}{8D}. \tag{9.19}$$

In order to apply the obtained inequalities to the estimate of the integral I_N from below, we represent it in the form (see (9.13) and (9.14))

$$\begin{aligned} I_N &= \sum_{i=1}^{N} a_i b_i + \sum_{i=1}^{N} a_i \gamma_{i,N} + \sum_{i=N+1}^{\infty} a_i \gamma_{i,N} \\ &= \sum_{j=1}^{t}\Big(\sum_{i \in A_j^N} a_i b_i\Big) + \sum_{j=1}^{t}\Big(\sum_{i \in A_j^N} a_i \gamma_{i,N}\Big) + \sum_{i=N+1}^{\infty} a_i \gamma_{i,N}. \end{aligned}$$

Let $N \in \mathbb{N}$ be so large that for every $j = 1, 2, \ldots, t$, the following inequality holds:

$$\sum_{i \in A_j} a_i^2 \le \frac{9}{4}\sum_{i \in A_j^N} a_i^2. \tag{9.20}$$

Then for every $j = 1, 2, \ldots, t$, there exist numbers b_i $(i \in A_j^N)$ satisfying Conditions (9.12) such that the relation

$$\sum_{i \in A_j^N} a_i b_i = \frac{1}{D}\Big(\sum_{i \in A_j^N} a_i^2\Big)^{1/2}$$

holds. Further, from relations (9.18), (9.19), and (9.10), we obtain the inequality

$$\begin{aligned} I_N \ \ge\ & \frac{1}{D}\sum_{j=1}^{t}\Big(\sum_{i \in A_j^N} a_i^2\Big)^{1/2} - \sum_{j=1}^{t}\Big(\sum_{i \in A_j^N} a_i^2\Big)^{1/2}\Big(\sum_{i \in A_j^N} \gamma_{i,N}^2\Big)^{1/2} \\ & - \Big(\sum_{i=N+1}^{\infty} |a_i|^2\Big)^{1/2}\Big(\sum_{i=N+1}^{\infty} \gamma_{i,N}^2\Big)^{1/2} \ \ge\ \frac{1}{2D}\sum_{j=1}^{t}\Big(\sum_{i \in A_j^N} a_i^2\Big)^{1/2}. \end{aligned}$$

The last inequality and relation (9.20) show that for every $t \in \mathbb{N}$ and arbitrary partition $\{A_j\}_{j=1}^t$ of the set of integers there exist large enough N and numbers b_i $(i = 1, 2, \ldots, N)$ satisfying Conditions (9.12) such that

$$I_N = \int\limits_{\Omega} R_N(\omega)\varphi(\omega)\, d\mathbb{P}(\omega) \ge \frac{1}{3D}\sum_{j=1}^{t}\left(\sum_{i \in A_j} a_i^2\right)^{1/2}. \tag{9.21}$$

The next step of the proof is to estimate the integral I_N from above by $C\|\varphi\|_t$, where $C > 0$ is a constant. Due to the Hölder inequality

$$|I_N| \le \|R_N\|_{t'}\, \|\varphi\|_t, \ \ t' = \frac{t}{t-1}, \tag{9.22}$$

and the non-negativity of Riesz products, it suffices to estimate the quantity

$$L_N = \ \|R_N\|_{t'}^{t'} = \int\limits_{\Omega} [R_N(\omega)]^{t'}\, d\mathbb{P}(\omega) \tag{9.23}$$

from above.

Let $t \in \mathbb{N}$, $t \geq 3$, $\{A_j\}_{j=1}^{t}$ be a partition of $\mathbb{N}$, $A_j^N = \{i = 1, \dots, N : i \in A_j\}$, and numbers b_i satisfy relations (9.12). From an elementary inequality $(1+x)^y \leq 1 + yx$ $(x \geq 1, 0 < y \leq 1)$, it follows that

$$\begin{aligned} \left[R_N(\omega)\right]^{t'} &\leq \prod_{i=1}^{N}(1 + b_i\varphi_i(\omega))\left[1 + (t-1)^{-1} b_i\varphi_i(\omega)\right] \\ &\leq \prod_{i=1}^{N}\left(1 + \frac{D^2}{t-1} b_i^2 + \frac{t}{t-1} b_i\varphi_i(\omega)\right) \\ &= \prod_{j=1}^{t}\prod_{i\in A_j^N}\left(1 + \frac{D^2}{t-1} b_i^2 + \frac{t}{t-1} b_i\varphi_i(\omega)\right). \end{aligned} \tag{9.24}$$

If $|B| := \operatorname{card} B$, then the inner product in the last expression is equal to the sum

$$\begin{aligned} &1 + \frac{D^2}{t-1}\sum_{i\in A_j^N} b_i^2 + \left(\frac{D^2}{t-1}\right)^2 \sum_{\substack{i_1<i_2 \\ i_1,i_2\in A_j^N}} b_{i_1}^2 b_{i_2}^2 + \dots + \left(\frac{D^2}{t-1}\right)^{|A_j^N|}\prod_{i\in A_j^N} b_i^2 \\ &+ \sum_{\varnothing\neq C_j \overset{\neq}{\subset} A_j^N}\left\{1 + \left(\frac{D^2}{t-1}\right)^{|A_j^N|-|C_j|}\prod_{i\in A_j^N\setminus C_j} b_i^2\right\}\left(\frac{t}{t-1}\right)^{|C_j|}\prod_{i\in C_j} b_i\varphi_i(\omega) \\ &+ \left(\frac{t}{t-1}\right)^{|A_j^N|}\prod_{i\in A_j^N} b_i\varphi_i(\omega). \end{aligned}$$

Since, due to inequality (9.12), the inequalities

$$1 + \frac{D^2}{t-1}\sum_{i\in A_j^N} b_i^2 + \left(\frac{D^2}{t-1}\right)^2 \sum_{\substack{i_1<i_2 \\ i_1,i_2\in A_j^N}} b_{i_1}^2 b_{i_2}^2 + \dots + \left(\frac{D^2}{t-1}\right)^{|A_j^N|}\prod_{i\in A_j^N} b_i^2 \leq \frac{t-1}{t-2}$$

and

$$1 + \left(\frac{D^2}{t-1}\right)^{|A_j^N|-|C_j|}\prod_{i\in A_j^N\setminus C_j} b_i^2 \leq \frac{t}{t-1}$$

hold, it follows from relations (9.23), (9.24), and (9.12) that

$$\begin{aligned} L_N \leq{}& \left(\frac{t-1}{t-2}\right)^{t} + \left(\frac{t-1}{t-2}\right)^{t-1}\sum_{j=1}^{t}\sum_{C_j\subset A_j^N}\left(\frac{t}{t-1}\right)^{|C_j|+1}\left|\mathbb{E}\left(\prod_{i\in C_j}\varphi_i(\omega)\right)\right| \\ &+ \left(\frac{t-1}{t-2}\right)^{t-2}\sum_{1\leq j_1<j_2\leq t}\sum_{C_{j_1}\subset A_{j_1}^N}\sum_{C_{j_2}\subset A_{j_2}^N}\left(\frac{t}{t-1}\right)^{|C_{j_1}|+|C_{j_2}|+2}\left|\mathbb{E}\left(\prod_{k=1}^{2}\prod_{i\in C_{j_k}}\varphi_i(\omega)\right)\right| \\ &\dots + \left(\frac{t-1}{t-2}\right)^{t-s}\sum_{1\leq j_1<\dots<j_s\leq t}\sum_{C_{j_1}\subset A_{j_1}^N}\cdots\sum_{C_{j_s}\subset A_{j_s}^N}\left(\frac{t}{t-1}\right)^{\sum_{k=1}^{s}|C_{j_k}|+s}\left|\mathbb{E}\left(\prod_{k=1}^{s}\prod_{i\in C_{j_k}}\varphi_i(\omega)\right)\right| \\ &\dots + \sum_{C_1\subset A_1^N}\cdots\sum_{C_t\subset A_t^N}\left(\frac{t}{t-1}\right)^{\sum_{j=1}^{t}|C_j|+t}\left|\mathbb{E}\left(\prod_{j=1}^{t}\prod_{i\in C_j}\varphi_i(\omega)\right)\right|. \end{aligned} \tag{9.25}$$

Note that for $t \geq 3$, the following inequalities hold:

$$\frac{t}{t-1} \leq 2 \quad \text{and} \quad \left(\frac{t-1}{t-2}\right)^{t-s} \left(\frac{t}{t-1}\right)^{s} \leq 4e \ \ (s = 0, 1, \ldots, t). \tag{9.26}$$

Moreover, since sets A_j^N ($j = 1, 2, \ldots, t$) are pairwise disjoint, it follows that integrand functions on the right-hand side of (9.25) are different products of pairwise different functions φ_i ($i = 1, 2, \ldots, N$). Their number is equal to $|C_j|$ ($j = 1, 2, \ldots, t$) for summands of the first sum, $|C_{j_1}|+|C_{j_2}|$ ($1 \leq j_1 < j_2 \leq t$) for summands of the second, $\ldots$, and $\sum_{j=1}^{t} |C_j|$ for summands of the last sum. Therefore, the greatest index for functions φ_i in every integrand product is not less than $|C_j|$, $|C_{j_1}| + |C_{j_2}|, \ldots, \sum_{j=1}^{t} |C_j|$ respectively. Finally, from (9.25) and (9.26), we obtain the inequality

$$L_N \leq 4e \left\{ 1 + \sum_{i=1}^{N} 2^i \sum_{1 \leq j_1 < \ldots < j_s \leq i-1} \left|\mathbb{E}(\varphi_{j_1} \ldots \varphi_{j_s} \varphi_i)\right| \right\}, \tag{9.27}$$

where the inner sum is taken over all sets of indices $1 \leq j_1 < \ldots < j_s \leq i - 1$. Therefore, from relations (9.10) and (9.11) and from the fact that $D \geq 1$, it follows that

$$L_N \leq \frac{4e}{D}\Big(1 + \sum_{i=1}^{\infty} \varepsilon_i\Big) \ < \ 13;$$

therefore (see (9.22) and (9.23)), for all $N \in \mathbb{N}$ and $t = 3, 4, \ldots$, we have

$$|I_N| \leq 13\|\varphi\|_t.$$

The last inequality and relation (9.21) yield the inequality

$$\sum_{j=1}^{t} \Big(\sum_{i \in A_j} a_i^2 \Big)^{1/2} \leq 39D\|\varphi\|_t,$$

whence, due to (9.8) and (9.9) (see Lemma 2.2 as well), we obtain that for any integer $t \geq 3$ and arbitrary $a = (a_i)_{i=1}^{\infty} \in \ell_2$ the following inequality holds:

$$\kappa_a(\sqrt{t}) \leq 39\sqrt{2}\, D\|\varphi\|_t.$$

Using the concavity of the $\mathcal{K}$-functional, we see that the following inequality holds for every real $t \geq 3$:

$$\kappa_a(\sqrt{t}) \leq 78\, D\|\varphi\|_t. \tag{9.28}$$

We have to show that a similar inequality holds if $1 \leq t \leq 3$. Since $\|a\|_2 \leq \|a\|_1$, it follows that $\mathcal{K}(1, a; \ell_1, \ell_2) = \|a\|_2$; therefore, due to the concavity of the $\mathcal{K}$-functional, it suffices to prove that there exists a constant $M = M(D)$ such that

$$\|a\|_2 \leq M \,\Big\|\sum_{i=1}^{\infty} a_i\varphi_i\Big\|_1 \ \ (a = (a_i)_{i=1}^{\infty} \in \ell_2). \tag{9.29}$$

First of all, since $\|\varphi_i\|_2 = 1$, it follows from (9.10), (9.11), and the Cauchy–Bunyakovskii inequality that for arbitrary $m \in \mathbb{N}$, the following inequality holds:

$$\Big\|\sum_{i=1}^{m} a_i\varphi_i\Big\|_2^2 = \sum_{i=1}^{m} a_i^2 + 2 \sum_{1 \leq i < j \leq m} a_i a_j \mathbb{E}(\varphi_i\varphi_j)$$

$$\geq \sum_{i=1}^{m} a_i^2 \left\{ 1 \ - \ 2\Big[\sum_{1 \leq i < j \leq m} (\mathbb{E}(\varphi_i\varphi_j))^2 \Big]^{1/2} \right\} \geq \frac{1}{2}\|(a_i)\|_2^2.$$

Then, since the sequence $\{f_n\}$ satisfies the assumptions of Theorem 9.2, it follows that we can assume that the inequality

$$\Big\|\sum_{k=1}^{\infty} b_k\varphi_k\Big\|_p \le 2D\Big\|\sum_{k=1}^{\infty} b_k r_k\Big\|_p$$

is valid for arbitrary $(b_k) \in \ell_2$ and $p \ge 1$ (see relation (9.6)). Applying the last two inequalities and using the Khintchine inequality (as in the proof of Theorem 1.4), we obtain the inequality

$$\|a\|_2^2 \le 2\Big\|\sum_{i=1}^{m} a_i\varphi_i\Big\|_2^2 \le 2\Big(\int_0^1\Big|\sum_{i=1}^{m} a_i\varphi_i(t)\Big|\,dt\Big)^{2/3}\Big(\int_0^1\Big(\sum_{i=1}^{m} a_i\varphi_i(t)\Big)^4 dt\Big)^{1/3}$$

$$\le 2\cdot 2^{4/3}D^{4/3}\Big\|\sum_{i=1}^{m} a_i\varphi_i\Big\|_1^{2/3}\Big(\int_0^1\Big(\sum_{i=1}^{m} a_i r_i(t)\Big)^4 dt\Big)^{1/3} \le 8\cdot D^{4/3}\Big\|\sum_{i=1}^{m} a_i\varphi_i\Big\|_1^{2/3}\|a\|_2^{4/3},$$

whence

$$\|a\|_2 \le 16\sqrt{2}D^2\Big\|\sum_{i=1}^{m} a_i\varphi_i\Big\|_1.$$

Finally, inequality (9.29) holds with $M = 16\sqrt{2}D^2$. As was mentioned, this and inequality (9.28) imply that (9.7) holds for every $t \ge 1$ with a constant depending on D and d. Therefore, the theorem is proved. □

Proof of Theorem 9.1. Assume that $\{f_n\}_{n=1}^{\infty}$ contains a subsequence satisfying Conditions (1)–(3). Then the existence of a subsequence $\{\varphi_i\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ equivalent in distribution to the Rademacher system is a corollary from Theorems 9.2, 9.3, and 8.2. Indeed, applying Theorems 9.2, 9.3 we extract $\{\varphi_i\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that

$$\Big\|\sum_{i=1}^{\infty} a_i\varphi_i\Big\|_t \asymp \kappa_a(\sqrt{t}),\quad a \in \ell_2,\ t \ge 1.$$

Then due to Theorem 8.2, the sequence $\{\varphi_i\}_{i=1}^{\infty}$ is equivalent in distribution to the Rademacher system. Conversely, let there exist sequence $\{\varphi_i\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ equivalent in distribution to the Rademacher system. Then, from the definition of the equivalence in distribution and the fact that the Rademacher system has properties (1) and (3), we obtain that $\{\varphi_i\}_{i=1}^{\infty}$ possesses these properties as well. It remains to show that some subsequence $\{\varphi_i\}$ weakly converges to zero in L_2.

Since $\{\varphi_i\}$ is bounded in L_2, it follows from the weak relative compactness of bounded sets in this space that there exist $\{\varphi_k'\} \subset \{\varphi_i\}$ and $f \in L_2$ such that $\varphi_k' \to f$ weakly in L_2. As was mentioned, $\{\varphi_i\}$ is uniformly bounded, i.e., $|\varphi_i(\omega)| \le D$ with some $D > 0$ and all $i \in \mathbb{N}$ and $\omega \in \Omega$. One can easily check that a similar inequality holds for the weak limit as well, i.e., $|f(\omega)| \le D$ ($\omega \in \Omega$). Therefore, if $\psi_k := \varphi_k' - f$, then $|\psi_k(\omega)| \le 2D$ for $k \in \mathbb{N}$ and $\omega \in \Omega$. Moreover, since $\{\psi_k\}$ weakly converges to zero in L_2, it follows that we can apply Corollary 9.1 to find a subsequence $\{\psi_{k_j}\} \subset \{\psi_k\}$ such that $\{\psi_{k_j}\} \overset{\mathbb{P}}{\prec} \{r_k\}$. Therefore, for arbitrary $n \in \mathbb{N}$, $a_j \ge 0$ ($j = 1, 2, \ldots, n$), and $p > 2$, the following inequality holds:

$$\sum_{j=1}^{n} a_j\|f\|_p \le \Big\|\sum_{j=1}^{n} a_j\varphi_{k_j}'\Big\|_p + \Big\|\sum_{j=1}^{n} a_j\psi_{k_j}\Big\|_p \le C\Big\|\sum_{j=1}^{n} a_j r_j\Big\|_p \le C_1\Big(\sum_{j=1}^{n} a_j^2\Big)^{1/2},$$

where C_1 depends on p and D, but does not depend on $n \in \mathbb{N}$. Therefore, $f = 0$ and the proof is finished. □

Since Conditions (1)–(3) of Theorem 9.1 hold for uniformly bounded orthonormal systems, we have the following important statement.

Theorem 9.4. *If $\{f_n\}_{n=1}^{\infty}$ is an orthonormal sequence of random variables defined on a probability space $(\Omega, \Sigma, \mathbb{P})$, $|f_n(\omega)| \le D$ $(n = 1, 2, ..; \omega \in \Omega)$, then we can extract a subsequence $\{\varphi_i\}_{i=1}^{\infty}$ equivalent in distribution to the Rademacher system. The constant of this equivalence depends only on D.*

Note that the Rademacher system is in a sense the best lacunary sequence: by Theorem 1.3, it is an S_∞-system with constants $K_p \asymp \sqrt{p}$ $(p \to \infty)$ and, due to Remark 1.5, it is a Sidon system (see definitions in Sec. 0.4). It is obvious that any sequence of random variables equivalent in distribution to the Rademacher system has the same properties. The results obtained give us exact conditions ensuring that uniformly bounded systems of functions contain strongly lacunary subsequences. This shows that a subsequence has at the same time different lacunary properties and shows the character of these properties. For example, show that the "sidonity" of a sequence $\{\varphi_i\}_{i=1}^{\infty}$ equivalent in distribution to the Rademacher system is strong in the sense that the knowledge of that order of the polynomial $\sum\limits_{i=1}^{m} a_i\varphi_i$ allows us to control the measure of the set, where its absolute value is more than $\sum\limits_{i=1}^{m} |a_i|$ (up to a constant).

Corollary 9.4. *Assume that a system of random variables $\{f_n\}_{n=1}^{\infty}$ contains a subsequence $\{f_{n_k}\}$ satisfying Conditions* (1)–(3) *of Theorem* 9.1. *Then $\{f_n\}_{n=1}^{\infty}$ contains a subsystem $\{\varphi_i\}_{i=1}^{\infty}$ possessing the following property*: *There exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$ depending only on D and d such that for any polynomial*

$$T(\omega) = \sum_{i=1}^{m} a_i\varphi_i(\omega)$$

there exists a set $E = E(T) \subset \Omega$, $\mathbb{P}(E) > \alpha_1 2^{-m}$ such that

$$|T(\omega)| \ge \alpha_2 \|T\|_\infty \ge \alpha_3 \sum_{i=1}^{m} |a_i| \quad (\omega \in E).$$

Proof. By Theorem 9.1, there exist a subsequence $\{\varphi_i\} \subset \{f_n\}$ and $C > 0$ such that

$$C^{-1}m(\{|\tilde{T}(x)| > Cz\} \le \mathbb{P}\{|T(\omega)| > z\} \le Cm\{|\tilde{T}(x)| > C^{-1}z\}$$

for all $z > 0$, where $\tilde{T}(x) = \sum\limits_{i=1}^{m} a_i r_i(x)$ $(x \in [0, 1])$. In particular, this yields the inequality

$$C^{-1}\|T\|_\infty \le \|\tilde{T}\|_\infty \le C\|T\|_\infty.$$

Therefore, from the definition of Rademacher functions, it follows that

$$\begin{aligned}\mathbb{P}\{|T(\omega)| > (2C^2)^{-1}\|T\|_\infty\} &\ge \mathbb{P}\{|T(\omega)| > (2C)^{-1}\|\tilde{T}\|_\infty\}\\ &\ge C^{-1}\, m\{|\tilde{T}(x)| > 2^{-1}\|\tilde{T}\|_\infty\} \ge C^{-1}2^{-m}.\end{aligned}$$

Assume that $\alpha_1 = C^{-1}$, $\alpha_2 = (2C^2)^{-1}$ and $E = E(T) = \{\omega \in \Omega:\ |T(\omega)| > \alpha_2\|T\|_\infty\}$. Then

$$|T(\omega)| > \alpha_2\|T\|_\infty \ge \alpha_2 C^{-1}\|\tilde{T}\|_\infty = \alpha_3 \sum_{i=1}^{m} |a_i|$$

for $\omega \in E$, where $\alpha_3 = \alpha_2 C^{-1}$. □

Recall (see Proposition 1.2) that the Rademacher system is a Sidon–Zygmund system in the narrow sense. The following result shows that there are conditions for a sequence of functions on $[0, 1]$ providing the existence of a subsequence everywhere "locally" equivalent in distribution to the Rademacher system (and, therefore, possessing the Sidon–Zygmund property in the narrow sense).

Theorem 9.5. *Assume that a sequence $\{f_n\}_{n=1}^{\infty}$ of functions measurable on $[0,1]$ contains a subsequence $\{f_{n_k}\}$ satisfying Conditions* (1) *and* (2) *of Theorem* 9.1 *and the following condition*:

$$d(I) := \inf_{k\to\infty} \int\limits_I f_{n_k}^2(x)\,dx > 0 \quad \textit{for any interval} \quad I \subset [0,1]. \tag{9.30}$$

Then there exists a subsequence $\{\varphi_i\} \subset \{f_n\}$ such that for every interval $I \subset [0,1]$ there exists a number $k_0 = k_0(I)$ such that the sequence $\{\varphi_i \chi_I\}_{i=k_0}^{\infty}$ is equivalent in distribution to the Rademacher system on $[0,1]$.

Proof. Due to Theorem 8.2, it suffices to find a subsystem $\{\varphi_i\} \subset \{f_n\}$ such that for every interval $I \subset [0,1]$ there exists $k_0 = k_0(I)$ such that

$$\Big\| \sum_{i=1}^{\infty} a_i \varphi_{k_0+i-1} \chi_I \Big\|_t \asymp \kappa_a(\sqrt{t}) \ \ (a \in \ell_2,\ t \geq 1) \tag{9.31}$$

(the equivalence constant depends on D and I).

First of all, by Theorem 9.2, there exists $\{g_i\} \subset \{f_{n_k}\}$ such that for any interval $I \subset [0,1]$ the following inequality holds:

$$\Big\| \sum_{i=1}^{\infty} a_i g_i \chi_I \Big\|_t \leq C\kappa_a(\sqrt{t}) \ \ (a \in \ell_2\,,\ t \geq 1), \tag{9.32}$$

where the constant $C > 0$ depends only on D and $d := d([0,1])$. Therefore, it remains to find only a subsystem $\{\varphi_i\} \subset \{g_i\}$ such that the inverse relation holds. First, show that for any function $g \in L_2$ and arbitrary $\varepsilon > 0$ there exists i_0 such that for any interval $I \subset [0,1]$ and all $i \geq i_0$ the following inequality holds:

$$\Big| \int\limits_I g(x) g_i(x)\,dx \Big| < \varepsilon. \tag{9.33}$$

For this function g, select an integer $m = m(g,\varepsilon)$ such that

$$\omega\Big(\frac{1}{m}\Big) := \sup_{E\subset[0,1],m(E)\leq 1/m} \Big(\int\limits_E g(x)^2\,dx \Big)^{1/2} < \frac{\varepsilon}{3D}.$$

Divide $[0,1]$ into m intervals $J_1, \dots, J_m$ of equal length. Then, due to the weak convergence of $\{g_i\}$ to zero, there exists i_0 such that

$$\sum_{k=1}^{m} \Big| \int\limits_{J_k} g(x) g_i(x)\,dx \Big| < \frac{\varepsilon}{3} \ \ (i \geq i_0).$$

Any interval $I \subset [0,1]$ is represented as a union of some intervals J_k and at most two intervals J' and J'' of length less than $1/m$. Therefore, from the last estimates and the Cauchy–Bunyakovskii inequality, it follows that

$$\Big| \int\limits_I g(x) g_i(x)\,dx \Big| \leq \frac{\varepsilon}{3} + \Big| \int\limits_{J'} g(x) g_i(x)\,dx \Big| + \Big| \int\limits_{J''} g(x) g_i(x)\,dx \Big| \leq \frac{\varepsilon}{3} + 2\omega\Big(\frac{1}{m}\Big) D < \varepsilon$$

if $i \geq i_0$. Therefore, (9.33) is proved.

Further, let (ε_i) be the same sequence as in the beginning of the proof of Theorem 9.3. Using (9.33) and arguing in the same way as in the proof of Theorem 9.3, we can extract a subsequence $\{\varphi_i\} \subset \{g_i\}$

satisfying (9.11) on every interval $I \subset [0,1]$. Moreover, we can choose a number $k_0 = k_0(I)$ such that, due to Condition (9.30), we have the relation

$$\int_I \varphi_i^2(x)\,dx \geq \frac{d(I)^2}{2} \quad \text{for all} \quad i \geq k_0$$

and, arguing similarly to the proof of Theorem 9.3 (in particular, using the fact that ε_i tends to zero "very" fast as $i \to \infty$), we can ensure that the inequality

$$\kappa_a(\sqrt{t}) \leq C_1 \Big\| \sum_{i=1}^{\infty} a_i \varphi_{k_0+i-1} \chi_I \Big\|_t \quad (a \in \ell_2\,,\, t \geq 1),$$

where the constant $C_1 > 0$ depends only on D and $d(I)$, holds. This and (9.32) imply (9.31), and the theorem is proved. □

Consider the problem of extracting lacunary subsystems from finite uniformly bounded orthonormal sets $\{f_n\}_{n=1}^N$ of functions defined on a segment $[0,1]$. We show that any such set contains a subsystem $\{f_{n_i}\}_{i=1}^s$ of "logarithmic density" (i.e., $s \geq C \log_2 N$) equivalent in distribution to a system s of Rademacher functions with a constant not depending on s. Begin from an exact wording of the main result.

Theorem 9.6. *Let $\{f_n\}_{n=1}^N$ be an orthonormal set of functions defined on $[0,1]$, $|f_n(x)| \leq D$, $n = 1, 2, \ldots, N$. Then there exist a set $\{f_{n_i}\}_{i=1}^s$, $1 \leq n_1 < n_2 < \ldots < n_s \leq N$, where $s \geq \max\{[1/7 \log_2 N], 1\}$, and $C > 0$ depending only on D such that for all $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, s$, and $z > 0$, the following inequality holds:*

$$C^{-1} m\Big\{\Big|\sum_{i=1}^{s} a_i r_i(x)\Big| > Cz\Big\} \leq m\Big\{\Big|\sum_{i=1}^{s} a_i f_{n_i}(x)\Big| > z\Big\} \leq C m\Big\{\Big|\sum_{i=1}^{s} a_i r_i(x)\Big| > C^{-1} z\Big\}.$$

To prove it, we need two lemmas. The first one shows that the "thinning" can yield systems with arbitrarily small mixed moments even for finite function sets.

Lemma 9.2. *From any set of functions $\{f_n\}_{n=1}^N$ ($N \geq 2^7$) satisfying assumptions of Theorem 9.6, we can extract functions $\{f_{n_i}\}_{i=1}^s$ ($1 \leq n_1 < n_2 < \ldots < n_s \leq N$) where $s \geq [1/7 \log_2 N]$, such that*

$$\sum_{\theta \in \mathcal{A}_s} \left\{ \mathbb{E} \left[\prod_{i=1}^{s} \left(\frac{f_{n_i}(x)}{D} \right)^{\theta_i} \right] \right\}^2 \leq 10^{-s}, \tag{9.34}$$

where $\mathcal{A}_s$ is the union of all collections $\theta = (\theta_i)_{i=1}^s$ such that

(a) $\theta_i = 0, 1$ *or* 2;
(b) $\sum_{i:\,\theta_i=1} 1 \geq 1$;
(c) $\sum_{i:\,\theta_i=2} 1 \leq 1$.

Proof. For a given number $s \leq N$ denote by $\mathcal{E}_N^s$ the union of all collections of integers $\{n_k\}_{k=1}^s$, $1 \leq n_1 < n_2 < \ldots < n_s \leq N$. The idea of the combinatorial proof is to estimate the average of sums

$$I(\{n_k\}) := \sum_{\theta \in \mathcal{A}_s} \left\{ \mathbb{E} \left[\prod_{i=1}^{s} \left(\frac{f_{n_i}(x)}{D} \right)^{\theta_i} \right] \right\}^2$$

over all $\{n_k\}_{k=1}^s \in \mathcal{E}_N^s$. First of all, note that all summands of the sum

$$G := \sum_{\{n_k\} \in \mathcal{E}_N^s} I(\{n_k\}) \tag{9.35}$$

can be represented as

$$\left(\frac{1}{D^j}\int_0^1 \prod_{k=1}^{j} f_{n_k}(x)\,dx\right)^2, \quad 1\le j\le s,\ \{n_k\}\in\mathcal{E}_N^j, \tag{9.36}$$

or

$$\left(\frac{1}{D^{j+1}}\int_0^1 f_{n_q}(x)\prod_{k=1}^{j} f_{n_k}(x)\,dx\right)^2, \quad 1< j\le s,\ 1\le q\le j,\ \{n_k\}\in\mathcal{E}_N^j. \tag{9.37}$$

Estimate sums

$$G_j' := \sum_{\{n_k\}\in\mathcal{E}_N^j}\left(\int_0^1 \prod_{k=1}^{j} f_{n_k}(x)\,dx\right)^2, \quad j=1,2,\dots,s,$$

and

$$G_j'' := \sum_{\{n_k\}\in\mathcal{E}_N^j}\sum_{q=1}^{j}\left(\int_0^1 f_{n_q}(x)\prod_{k=1}^{j} f_{n_k}(x)\,dx\right)^2, \quad j=1,2,\dots,s.$$

Since the system is orthonormal, it follows from the Bessel inequality that

$$G_1' := \sum_{n=1}^{N}\left(\int_0^1 f_n(x)\,dx\right)^2 \le \|\chi_{[0,1]}\|_2^2 = 1. \tag{9.38}$$

If $j>1$, then

$$G_j' \le \sum_{\{p_k\}\in\mathcal{E}_N^{j-1}}\ \sum_{\{n_k\}\in\mathcal{E}_N^j}\delta(\{p_k\},\{n_k\})\left(\int_0^1 \prod_{k=1}^{j} f_{n_k}(x)\,dx\right)^2,$$

where $\delta(\{p_k\},\{n_k\})$ is equal to 1 if $\{p_k\}\subset\{n_k\}$, and 0 otherwise. From the Bessel inequality, the following inequality holds for every collection $\{p_k\}\in\mathcal{E}_N^{j-1}$:

$$\sum_{\{n_k\}\in\mathcal{E}_N^j:\{n_k\}\supset\{p_k\}}\left(\int_0^1 \prod_{k=1}^{j} f_{n_k}(x)\,dx\right)^2 \le \sum_{n=1}^{N}\left\{\int_0^1\left(\prod_{k=1}^{j-1} f_{p_k}(x)\right) f_n(x)\,dx\right\}^2 \le \left\|\prod_{k=1}^{j-1} f_{p_k}\right\|_2^2 \le D^{2(j-1)}.$$

Therefore, the following inequality holds:

$$G_j' \le \sum_{\{p_k\}\in\mathcal{E}_N^{j-1}} D^{2(j-1)} = C_N^{j-1}D^{2(j-1)}, \quad j=2,3,\dots,s, \tag{9.39}$$

where C_m^l is the binomial coefficient.

To estimate the sum G_j'', we represent it in the form

$$G_j'' = \sum_{p=1}^{N} G_j''(p),$$

where

$$G_j''(p) = \sum_{\{n_k\}\in\mathcal{E}_N^{j-1}:\ p\notin\{n_k\}}\left(\int_0^1 f_p(x)^2\prod_{k=1}^{j-1} f_{n_k}(x)\,dx\right)^2.$$

Estimating the sum $G_j''(p)$ as G_j', we get the inequality

$$G_j''(p) \le C_{N-1}^{j-2}D^{2j}, \quad j=2,3,\dots,s,$$

which yields the inequality

$$G_j'' \le N C_{N-1}^{j-2} D^{2j}, \quad j = 2, 3, \dots, s. \tag{9.40}$$

Note that, for $s > 1$, each of summands (9.36) and (9.37) occurs in sum (9.35) exactly C_{N-j}^{s-j} times. Therefore, taking into account that $D \ge 1$ and $G_1'' = 0$, from relations (9.38)–(9.40), we obtain the inequality

$$G \le \sum_{j=1}^{s} \frac{1}{D^{2j}} C_{N-j}^{s-j} (G_j' + G_j'') \le \sum_{j=1}^{s} C_{N-j}^{s-j} (C_N^{j-1} + N C_{N-1}^{j-2}).$$

Since the number of elements in $\mathcal{E}_N^s$ is equal to C_N^s, it follows from the last inequality that

$$\min_{\{n_k\} \in \mathcal{E}_N^s} I(\{n_k\}) \le \frac{G}{C_N^s} \le \sum_{j=1}^{s} \frac{C_{N-j}^{s-j}}{C_N^s} (C_N^{j-1} + N C_{N-1}^{j-2}).$$

One can easily check that $j C_N^{j-1} \ge N C_{N-1}^{j-2}$; therefore, the right-hand side of the last relation is not greater than

$$(1+s) \sum_{j=1}^{s} \frac{C_{N-j}^{s-j} C_N^{j-1}}{C_N^s} = (1+s) \sum_{j=1}^{s} \frac{1}{N-j+1} \cdot \frac{s!}{(s-j)!(j-1)!}$$

$$\le \frac{(1+s)}{N-s} \sum_{j=1}^{s} j C_s^j \le \frac{2^s s(1+s)}{N-s} \le 10^{-s}$$

if $N \ge (128)^s$, i.e., $s \le \frac{1}{7} \log_2 N$. The lemma is proved. □

The second lemma allows us to extend (under certain conditions) a uniformly bounded set of functions defined on $[0,1]$ to a wider set such that it becomes a multiplicative uniformly bounded set.

Lemma 9.3. *Let $\{g_i\}_{i=1}^s$ be a set of functions on $[0,1]$, $|g_i(x)| \le D$ for all $i = 1, 2, \dots, s$ and $x \in [0,1]$, and*

$$\max_{\theta \in \mathcal{A}_s'} \left| \mathbb{E}\left[\prod_{i=1}^{s} \left(\frac{g_i(x)}{D} \right)^{\theta_i} \right] \right| \; < \; 2^{-s}, \tag{9.41}$$

where $\mathcal{A}_s'$ is the union of all collections $\theta = (\theta_i)_{i=1}^s \in \mathcal{A}_s$ such that $\theta_i = 0$ or 1.

Then there exists a multiplicative set $\{h_i\}_{i=1}^s$ of functions defined on the segment $[0,2]$ such that $h_i(x)\chi_{[0,1]}(x) = g_i(x)$ and $\|h_i\|_{L_\infty[0,2]} \le D$ $(i = 1, 2, \dots, s)$.

Proof. Consider the representation

$$[1,2) = \bigcup_{k=1}^{2^s} \Delta_k \,, \; \Delta_k = [a_{k-1}, a_k) \,, \; a_k = 1 + k 2^{-s} \, (k = 0, 1, 2, \dots, 2^s).$$

Since $\operatorname{card} \mathcal{A}_s' = 2^s - 1$, it follows that we can establish a one-to-one correspondence between the set $\mathcal{A}_s'$ and the family of intervals $\{\Delta_k\}_{k=1}^{2^s-1}$. Take one of such correspondences (arbitrarily) and fix it.

Let $\theta = (\theta_i)_{i=1}^s$ be the element of $\mathcal{A}_s'$ corresponding to Δ_k $(1 \le k \le 2^s - 1)$. Introduce $\{i_j\}_{j=1}^m = \{i = 1, 2, \dots, s : \theta_i = 1\}$, $i_1 < i_2 < \dots < i_m$ $(m = 1, 2, \dots, s)$. We want to find extensions h_i of the functions g_i on $[0,2]$ satisfying the following relations:

$$\int_{\Delta_k} h_{i_1} h_{i_2} \dots h_{i_m} \, dx = -\mathbb{E}(g_{i_1} g_{i_2} \dots g_{i_m}) \tag{9.42}$$

and

$$\int_{\Delta_k} h_{i_1'} h_{i_2'} \dots h_{i_l'} \, dx = 0 \tag{9.43}$$

for any collection $\{i'_j\}_{j=1}^l$ not corresponding (as an element of $\mathcal{A}_s{}'$) to Δ_k.

For arbitrary $\alpha \in \Delta_k$, define a function

$$u_\alpha(x) = \chi_{[a_{k-1},\alpha)}(x) - \chi_{[\alpha,a_k]}(x) \ \ (x \in \Delta_k).$$

Then the function

$$v(\alpha) = \int\limits_{\Delta_k} u_\alpha(x)\,dx = 2\alpha - a_{k-1} - a_k \ \ (a_{k-1} \le \alpha \le a_k)$$

takes all values from a segment $[-2^{-s}, 2^{-s}]$. By Condition (9.41), the following inequality holds for functions $\bar{g}_i(x) = g_i(x)/D : |\mathbb{E}(\bar{g}_{i_1}\bar{g}_{i_2}\ldots\bar{g}_{i_m})| < 2^{-s}$. Then there exists $\alpha_k \in (a_{k-1}, a_k)$ such that

$$v(\alpha_k) = -\mathbb{E}(\bar{g}_{i_1}\bar{g}_{i_2}\ldots\bar{g}_{i_m}). \tag{9.44}$$

First, let $m \ge 2$. Take a system of step functions $d_j(x)$, $x \in [a_{k-1}, \alpha_k)$ $(j = 1, 2, \ldots, m-1)$ with the following properties:

$$|d_j(x)| \equiv 1 \quad \text{and} \quad \int\limits_{a_{k-1}}^{\alpha_k} d_{j_1}d_{j_2}\ldots d_{j_l}\,dx = 0 \tag{9.45}$$

for arbitrary collections $1 \le j_1 < j_2 < \ldots < j_l \le m-1$ (e.g., such a system can be generated using Rademacher functions. Denote by $c_j(x)$, $x \in [\alpha_k, a_k]$ $(j = 1, 2, \ldots, m-1)$, the analogous system generated by $[\alpha_k, a_k]$.

Define functions $h_i(x)$ $(i = 1, 2, \ldots, s)$ for $x \in \Delta_k$ $(k = 1, \ldots, 2^s - 1)$. If $i \ne i_j$ for any $j = 1, 2, \ldots, m$ (recall that the collection $\{i_j\}_{j=1}^m$ corresponds to Δ_k as an element of $\mathcal{A}_s{}'$), then assume that $h_i(x) \equiv 0$. Otherwise, assume that

$$h_{i_j}(x) = D\left(d_j(x)\chi_{[a_{k-1},\alpha_k)}(x) + c_j(x)\chi_{[\alpha_k,a_k]}(x)\right) \quad \text{if} \quad j = 1, 2, \ldots, m-1$$

and

$$h_{i_m}(x) = D\Big(\prod_{j=1}^{m-1} d_j(x)\chi_{[a_{k-1},\alpha_k)}(x) - \prod_{j=1}^{m-1} c_j(x)\chi_{[\alpha_k,a_k]}(x)\Big).$$

Due to (9.45) and (9.44), we have

$$\int\limits_{\Delta_k} h_{i_1}h_{i_2}\ldots h_{i_m}\,dx = D^m \int\limits_{\Delta_k} u_{\alpha_k}(x)\,dx = D^m v(\alpha_k) = -\mathbb{E}(g_{i_1}g_{i_2}\ldots g_{i_m})$$

and we obtain (9.42). Note that (9.43) follows from the definition of functions h_i $(i = 1, 2, \ldots, s)$ and (9.45).

If $m = 1$, then we define h_i on Δ_k as follows:

$$h_i(x) \equiv 0, \quad \text{if} \quad i \ne i_1, \quad \text{and} \quad h_{i_1}(x) = Du_{\alpha_k}(x).$$

One can easily see that relations (9.42) and (9.43) hold for $m = 1$ as well.

Extend functions $h_i(x)$ to the whole segment $[0, 2] : h_i(x) = g_i(x)$ $(x \in [0, 1])$ and $h_i(x) = 0$ $(x \in \Delta_{2^s})$. Then, due to the definition and relations (9.42) and (9.43), the set $\{h_i\}_{i=1}^s$ is multiplicative on the segment $[0, 2]$ and $|h_i(x)| \le D$. $\square$

Proof of Theorem 9.6. Due to Theorem 8.2 and Remark 8.3, it is sufficient to show that there exists a set $\{f_{n_i}\}_{i=1}^s$, $s \ge \max\{[1/7\log_2 N], 1\}$, and a constant depending only on D such that

$$\Big\|\sum_{i=1}^s a_i f_{n_i}\Big\|_t \asymp \kappa_a(\sqrt{t}) \ \ (t \ge 1\,,\ a = (a_i)_{i=1}^s), \tag{9.46}$$

where $\kappa_a(t) = \mathcal{K}(t, a; \ell_1, \ell_2)$.

Using Lemma 9.2, extract a subsystem $\{f_{n_i}\}_{i=1}^s$ such that it satisfies Condition (9.34) and $s \geq \max\{[1/7\log_2 N], 1\}$. Since $\mathcal{A}_s{}' \subset \mathcal{A}_s$, it follows that relation (9.41) holds for $g_i = f_{n_i}$. Therefore, by Lemma 9.3, the set $\{f_{n_i}\}_{i=1}^s$ can be extended to a set $\{h_i\}_{i=1}^s$ multiplicative on $[0, 2]$, $|h_i(x)| \leq D$. Then functions $h_i'(x) = h_i(2x)$ form a system multiplicative on $[0, 1]$, $|h_i'(x)| \leq D$; therefore, due to Corollary 8.2 and Theorem 2.1, for arbitrary $t \geq 1$ and $a_i \in \mathbb{R}$ $(i = 1, \ldots, s)$, we have

$$\Big\| \sum_{i=1}^{s} a_i h_i' \Big\|_{L_t[0,1]} \leq C' \kappa_a(\sqrt{t}),$$

whence

$$\Big\| \sum_{i=1}^{s} a_i f_{n_i} \Big\|_t \leq C \kappa_a(\sqrt{t}), \tag{9.47}$$

where the constant $C > 0$ depends only on D.

To prove the inverse inequality, introduce

$$a = (a_i)_{i=1}^s \quad \text{and} \quad P(x) = \sum_{i=1}^{s} a_i f_{n_i}(x).$$

First, let $t \in \mathbb{N}$ and $t \geq 3$. Then, if $s \leq 16D^2$, it follows from the orthonormality of the system, the Cauchy–Bunyakovskii inequality, and the definition of the $\mathcal{K}$-functional that

$$\|P\|_t \geq \|P\|_2 = \|a\|_2 \geq \frac{1}{\sqrt{s}} \sum_{i=1}^{s} |a_i| \geq \frac{1}{4D} \kappa_a(\sqrt{t}). \tag{9.48}$$

Consider the more complicated case where

$$s \; > \; 16D^2. \tag{9.49}$$

Let $\{A_j\}_{j=1}^t$ be an arbitrary partition of a set $\{1, 2, \ldots, s\}$ into t disjoint subsets (some of them may be empty), and $(b_i)_{i=1}^s$ be a set of numbers satisfying the condition

$$\sum_{i \in A_j} b_i^2 \leq 1 \;\; (j = 1, 2, \ldots, t). \tag{9.50}$$

Construct the Riesz products

$$R(x) = \prod_{i=1}^{s} \Big(1 + \frac{b_i}{D} f_{n_i}(x) \Big)$$

and estimate

$$I_s := \int_0^1 P(x) R(x)\, dx$$

from below. Since $\|f_{n_k}\|_2 = 1$ $(k = 1, \ldots, s)$, it follows that I_s can be represented as follows:

$$I_s = D^{-1} \sum_{i=1}^{s} a_i b_i + \sum_{i=1}^{s} a_i \gamma_i, \tag{9.51}$$

where

$$\gamma_i = \int_0^1 f_{n_i}(x)\, dx + \int_0^1 f_{n_i}(x) \sum_{\theta \in \mathcal{A}_s{}'(i)} \prod_{k=1}^{s} \Big(\frac{b_k}{D} f_{n_k}(x) \Big)^{\theta_k} dx,$$

$$\mathcal{A}_s{}'(i) = \mathcal{A}_s{}' \setminus \{\theta^i\}, \; \theta^i = (\theta^i_j), \theta^i_i = 1, \theta^i_j = 0 \; (j \neq i).$$

Since $|b_i| \le 1$ $(i = 1, 2, \dots, s)$ due to (9.50), it follows that

$$\sum_{i=1}^{s} |\gamma_i| \le D \sum_{\theta \in \mathcal{A}_s} \left| \int_0^1 \prod_{k=1}^{s} \left(\frac{f_{n_k}(x)}{D} \right)^{\theta_k} dx \right|,$$

where $\mathcal{A}_s$ is a family of collections $\theta = (\theta_i)_{i=1}^s$ from Lemma 9.2. Note that $\operatorname{card} \mathcal{A}_s = 2^s - 1 + (2^{s-1} - 1)s \le s2^s$. Therefore, due to the Cauchy–Bunyakovskii inequality and Condition (9.34), we have

$$\sum_{i=1}^{s} |\gamma_i| \le D(s2^s)^{1/2} 10^{-s/2} = D(s5^{-s})^{1/2}.$$

Since $(s5^{-s})^{1/2} \le 8/s < 2^{-1}D^{-2}$ due to (9.49), it follows that

$$\sum_{i=1}^{s} |\gamma_i| \le \frac{1}{2D}. \tag{9.52}$$

For every $j = 1, 2, \dots, t$, we choose numbers b_i $(i \in A_j)$ such that (9.50) holds and

$$\sum_{i \in A_j} a_i b_i = \Big(\sum_{i \in A_j} a_i^2 \Big)^{1/2}.$$

Then, from (9.51) and (9.52), we obtain the inequality

$$\begin{aligned} I_s &\ge \frac{1}{D} \sum_{j=1}^{t} \big(\sum_{i \in A_j} a_i^2 \big)^{1/2} - \sum_{j=1}^{t} \big| \sum_{i \in A_j} a_i \gamma_i \big| \ge \frac{1}{D} \sum_{j=1}^{t} \big(\sum_{i \in A_j} a_i^2 \big)^{1/2} \\ &- \sum_{j=1}^{t} \big(\sum_{i \in A_j} a_i^2 \big)^{1/2} \big(\sum_{i \in A_j} |\gamma_i| \big) \ge \frac{1}{2D} \sum_{j=1}^{t} \big(\sum_{i \in A_j} a_i^2 \big)^{1/2}. \end{aligned}$$

Since the partition $\{A_j\}_{j=1}^t$ is taken arbitrarily, it follows from (9.8) that

$$I_s \ge \frac{1}{2\sqrt{2}D} \, \kappa_a(\sqrt{t}) \quad (t \in \mathbb{N}, t \ge 3). \tag{9.53}$$

Now, estimate I_s from above. First of all, by the Hölder's inequality, we have

$$|I_s| \le \|R\|_{t'} \|P\|_t \,, \quad t' = \frac{t}{t-1}. \tag{9.54}$$

Then, as in the proof of Theorem 9.3 (see (9.27)), we can show that the following relation holds for $t \ge 3$:

$$\|R\|_{t'}^{t'} \le 4e \left(1 + 2^s \sum_{\theta \in \mathcal{A}_s'} \left| \mathbb{E} \left\{ \prod_{i=1}^{s} \left[\frac{f_{n_i}(x)}{D} \right]^{\theta_i} \right\} \right| \right).$$

Therefore, due to the Cauchy–Bunyakovskii inequality, the embedding ${\mathcal{A}_s}' \subset \mathcal{A}_s$, and Condition (9.34), we obtain the inequality

$$\|R\|_{t'} \le 4e(1 + 2^{3s/2} \cdot 10^{-s/2}) \le 8e.$$

Therefore, from (9.53) and (9.54), it follows that

$$\kappa_a(\sqrt{t}) \le 16\sqrt{2} e D \|P\|_t \quad (t \in \mathbb{N}, t \ge 3).$$

Similar relations hold for $t = 2$ and $t = 1$. Indeed, since $\kappa_a(1) = \|a\|_2$ and the system $\{f_n\}_{n=1}^N$ is orthonormal, it follows that

$$\kappa_a(\sqrt{2}) \le \sqrt{2} \kappa_a(1) = \sqrt{2} \|P\|_2 \quad \text{and} \quad \kappa_a(1) = \|a\|_2 \le \sqrt{2} \|P\|_1$$

where we use also the refined Khintchine L_1-inequality (see Theorem 1.4 and Remark 1.4). Extending the last estimates to the case of arbitrary real $t \geq 1$, we obtain the inequality

$$\kappa_a(\sqrt{t}) \leq C\|P\|_t \ \ (t \geq 1)$$

where C depends only on D. Combining the last relation with (9.47), we get (9.46). Its constants depend only on D. $\square$

Remark 9.1. As in the case of infinite systems, a set of functions $\{\varphi_i\}_{i=1}^s$ satisfying the assumption of Theorem 9.6 is both a Sidon system and an S_∞-system, and constants in the corresponding inequalities depend only on D (but not on the number of functions N in the initial set). Moreover, extraction theorems proved in this chapter cannot be improved in the sense that further "thinning" cannot improve distributions of corresponding polynomials. Indeed, due to Corollary 1.7, for any subsequence $\{r_{n_i}\}_{i=1}^\infty$ of the Rademacher system and arbitrary $m \in \mathbb{N}$, $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m)$ and $z > 0$, the following inequality holds:

$$m\Big\{\Big|\sum_{i=1}^m a_i r_{n_i}(x)\Big| > z\Big\} = m\Big\{\Big|\sum_{i=1}^m a_i r_i(x)\Big| > z\Big\}.$$

Therefore, any subsystem of a system equivalent in distribution to Rademacher system is again still equivalent to it.

Finally, we show that the result of Theorem 9.6 cannot be improved with respect to order. We prove this even for the Sidon property, which is weaker than the equivalence in distribution to the Rademacher system.

Proposition 9.1. *If a sequence $\{\sqrt{2}\cos 2\pi n_k x\}_{k=1}^\infty$ $(0 \leq x \leq 1)$ is a Sidon system, then there exists $C > 0$ such that*

$$\sum_{k:\, n_k \leq N} 1 \leq C \ln N \ \ (N = 2, 3, \ldots). \tag{9.55}$$

Proof. Let $f_n(x) := \sqrt{2}\cos 2\pi n x$ $(n = 1, 2, \ldots)$. Prove the validity of the following relation for arbitrary trigonometric polynomials:

$$m\Big\{t \in [0,1] : \ \Big\|\sum_{n=1}^N a_n f_n r_n(t)\Big\|_\infty > 8\Big(\sum_{n=1}^N a_n^2\Big)^{1/2} \ln^{1/2} N\Big\} \leq \frac{C}{N^2}, \tag{9.56}$$

where $C > 0$ is a universal constant. First, we show that for L_∞-norms of trigonometric polynomials $P(x) = \sum\limits_{n=1}^N a_n f_n(x)$, the following estimate holds:

$$\|P\|_\infty \leq 2 \max_{k=0,1,\ldots,36N^2} \Big|P\Big(\frac{k}{36N^2}\Big)\Big|. \tag{9.57}$$

Indeed, since the system $\{f_n\}_{n=1}^\infty$ is orthonormal, it follows from the Cauchy–Bunyakovskii inequality that

$$|P'(x)| \leq 2\sqrt{2}\pi N \sum_{n=1}^N |a_n| \leq 2\sqrt{2}\pi N^{3/2}\Big(\sum_{n=1}^N a_n^2\Big)^{1/2} \leq 2\sqrt{2}\pi N^{3/2}\|P\|_\infty \ \ (0 \leq x \leq 1).$$

Therefore, if x_0 from $[(k-1)/(36N^2), k/(36N^2)]$ $(k = 0, 1, \ldots, 36N^2)$ is such that $\|P\|_\infty = |P(x_0)|$, then

$$\Big|P(x_0) - P\Big(\frac{k}{36N^2}\Big)\Big| \leq \frac{1}{36N^2}\|P'\|_\infty \leq \frac{2\sqrt{2}\pi N^{3/2}}{36N^2}|P(x_0)| < \frac{1}{2}|P(x_0)|.$$

Therefore, $|P(k/(36N^2))| > \dfrac{1}{2}|P(x_0)| = \dfrac{1}{2}\|P\|_\infty$ and the relation (9.57) is proved.

Due to (9.57), we can estimate the left-hand side of relation (9.56) as follows:

$$m\Big\{t\in[0,1]:\ \Big\|\sum_{n=1}^{N}a_nf_nr_n(t)\Big\|_\infty>8\Big(\sum_{n=1}^{N}a_n^2\Big)^{1/2}\ln^{1/2}N\Big\}$$

$$\le m\Big\{t\in[0,1]:\ \max_{k=0,1,\dots,36N^2}\Big|\sum_{n=1}^{N}a_nf_n\Big(\frac{k}{36N^2}\Big)r_n(t)\Big|>4\Big(\sum_{n=1}^{N}a_n^2\Big)^{1/2}\ln^{1/2}N\Big\}$$

$$\le 36N^2\cdot\max_{k=0,1,\dots,36N^2}m\Big\{t\in[0,1]:\ \Big|\sum_{n=1}^{N}a_nf_n\Big(\frac{k}{36N^2}\Big)r_n(t)\Big|>4\Big(\sum_{n=1}^{N}a_n^2\Big)^{1/2}\ln^{1/2}N\Big\}.$$

Further, applying the following exponential estimate for distributions of Rademacher polynomials:

$$m\Big\{\Big|\sum_{n=1}^{N}b_nr_n\Big|>u\Big\}\le C'\exp\Big(-\frac{u^2}{2\sum\limits_{n=1}^{N}b_n^2}\Big)\quad(u>0)\tag{9.58}$$

(see Corollary 1.5), and the inequality $|f_n(x)|\le\sqrt{2}$ $(n=1,2,\dots)$, we obtain that the last value in the previous relation is not greater than

$$36N^2C'\max_{k=0,1,\dots,36N^2}\exp\left(-\frac{8\ln N\sum\limits_{n=1}^{N}a_n^2}{\sum\limits_{n=1}^{N}a_n^2f_n^2(k/(36N^2))}\right)\le 36N^2C'\exp(-4\ln N)=\frac{36C'}{N^2};$$

thus, relation (9.56) is proved.

Since the sequence $\{f_{n_k}\}$ is a Sidon system by assumption, it follows that for some C_1 and all $m\in\mathbb{N}$ $a_n\in\mathbb{R}$ $(n=1,2,\dots,m)$ the following inequality holds:

$$\sum_{k=1}^{m}|a_k|\le C_1\,\Big\|\sum_{k=1}^{m}a_kf_{n_k}\Big\|_\infty.$$

Let $a_n=0$ if $n\ne n_k$, $k=1,2,\dots$, and $a_{n_k}=1$ $(k=1,2,\dots)$. Then if the last relation does not contradict (9.56), then the inequality

$$\sum_{k:\,n_k\le N}1\le 8C_1\,\ln^{1/2}N\Big(\sum_{k:\,n_k\le N}1\Big)^{1/2}$$

holds, whence (9.55) holds and the proposition is proved. □

Comments and References. Extraction of lacunary subsystems is considered in detail by Gaposhkin in the survey [70]. Recall the most important results of [70] related to the integrability and absolute convergence of the series generated by a given system of functions. Due to the Khintchine inequality, the sum of a series $\sum\limits_{n=1}^{\infty}a_nr_n(x)$ $(x\in[0,1])$ belongs to all spaces L_p provided that $(a_n)_{n=1}^\infty\in\ell_2$. For the lacunary trigonometric series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos n_kx+b_k\sin n_kx,\quad\text{where}\quad\frac{n_{k+1}}{n_k}\ge q>1,\tag{9.59}$$

a similar result is obtained by Zygmund in [140]. Then, in the articles of Banach and Sidon, notions of a lacunary system of order p (short S_p-system) and an S_∞-system appeared. Banach proved that from any orthonormal sequence $\{f_n\}$ of measurable functions defined on $[0,1]$ such that $\limsup_{n\to\infty}\|f_n\|_p<\infty$, one can extract an S_p-subsystem (see [36] or [81, Sec. 7.2]). In particular, this implies a possibility to extract an S_∞-subsystem from any uniformly bounded orthonormal system of functions. The following (more exact) result is obtained in [70, Theorem 1.3.1]. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions measurable on $[0,1]$ and $p>2$. An S_p -subsystem can be extracted from it if and only if there exists a sequence $\{f_{n_k}\}$ such that $\|f_{n_k}\|_p \le D$ $(k=1,2,\dots)$ and $f_{n_k} \to 0$ weakly in L_2.

The study of absolute convergence of lacunary series begins with the article [68] by Fatou, where the fact that any everywhere convergent series (9.59), where $q>3$, converges absolutely is proved. Then the same property was obtained by Sidon for any lacunary series (9.59) defining a bounded function (see [130]). A local variant of this theorem is proved in [142] by Zygmund. Similar results for the Rademacher system and other particular lacunary systems of functions are obtained in [80]. The notion of the Sidon system and the Sidon–Zygmund system generalizing these results appear. The following result is obtained in [70, Theorems 1.4.1 and 1.4.2]. Let a system $\{f_n\}_{n=1}^{\infty}$ of functions measurable on $[0,1]$ have the following properties:

(1) $\|f_n\|_2 = 1$ $(n=1,2,\dots)$;
(2) $|f_n(x)| \le D$ $(n=1,2,\dots;\, x\in[0,1])$;
(3) there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $f_{n_k} \to 0$ weakly in L_2.

Then from $\{f_n\}$ one can extract both a Sidon subsequence and a Sidon–Zygmund subsequence. If the subsequence from Condition (3) is such that

$$\liminf_{k\to\infty} \int\limits_F f_{n_k}^2(x)\,dx \;>\; 0 \text{ for every } F\subset[0,1],\ m(F)>0,$$

a Sidon–Zygmund subsystem in the narrow sense then can be extracted from $\{f_n\}$.

Theorems 9.1 and 9.4 and Corollaries 9.4 and 9.5, providing the lacunarity of the extracted subsystem in both senses as well as certain additional properties, strengthen and complement the given statements. All these results and Theorem 9.3, which plays the main role in the proof of Theorem 9.1, are obtained in the paper [18] by the author. Lemma 9.1 is proved in [15]. Its proof uses the idea of the proof of [71, Lemma A]. The assertion of Corollary 9.2 on the possibility of extracting exponentially integrable subsystems was announced in [70, Theorem 1.3.2]; the detailed proof is published in [15]. Note that Theorem 9.1 has interesting applications both for the study of the Rademacher system and for the study of geometrical properties of symmetric spaces (see [64]).

Now, we pass to the extracting of lacunary subsystems from finite sets of random variables. Assume that $\{f_n\}_{n=1}^{N}$ is a set of random variables. We search the greatest $s=s(N)$ with the following property: there exists a set $\{f_{n_i}\}_{i=1}^{s} \subset \{f_n\}_{n=1}^{N}$ satisfying the lacunarity condition with a constant not depending on N. The following density theorem for finite Sidon subsystems is proved by Kashin (see [85] or [86, Theorem 9.4.9]). Let $\{f_n\}_{n=1}^{N}$ be an orthonormal set of functions defined on $[0,1]$, $|f_n(x)| \le D$ $(n=1,\dots,N;\, x\in[0,1])$. Then there exists a subsystem $\{f_{n_i}\}_{i=1}^{s} \subset \{f_n\}_{n=1}^{N}$ $(1\le n_1<\ldots<n_s\le N)$ such that $s \ge \max\{[1/6\log_2 N],1\}$ and the following inequality holds for any $a_i\in\mathbb{R}$ $(i=1,\dots,s)$:

$$D^{-1}\Big\|\sum_{i=1}^{s} a_i f_{n_i}\Big\|_\infty \le \sum_{i=1}^{s}|a_i| \le 4D\Big\|\sum_{i=1}^{s} a_i f_{n_i}\Big\|_\infty.$$

A weaker result (for $s\ge C(D)\ln\ln N$) is proved earlier in [107]. Theorem 9.6 shows that any such set contains a subsystem $\{f_{n_i}\}_{i=1}^{s}$ of the same “logarithmic density” (i.e. $s\ge C\log_2 N$), which is equivalent in distribution to the corresponding system s of Rademacher functions with a constant not depending on N. After proving this theorem in the work [18] (this result was also presented in [16]) the author realized that earlier a similar result (with the equivalence of L_p-norms instead of the equivalence in distribution) was proved by Szarek in [135, Theorem 1]. The last result is formally weaker than Theorem 9.6. However, using Theorem 8.2 and Remark 8.3, we come to the conclusion that these statements are equivalent. Also, the following theorem about the density of subsystems majorizable by the Rademacher system in L_p for fixed $p>0$ is proved in [135, Theorem 2]. Let $\{f_n\}_{n=1}^{N}$ be an orthonormal set of functions defined on $[0,1]$, $|f_n(x)|\le D$ $(n=1,\dots,N;\, x\in[0,1])$. For arbitrary $p>0$, we assume the $p^*=4$ if $p<2$ and p^* is the least even integer number not less than p if $p\ge 2$.

Let $s \le N^{1/p^*}$. Then there exists a set $\{f_{n_i}\}_{i=1}^s \subset \{f_n\}_{n=1}^N$ $(1 \le n_1 < \ldots < n_s \le N)$ such that for any $a_i \in \mathbb{R}$ $(i = 1, \ldots, s)$ the following inequality holds:

$$\Big\|\sum_{i=1}^{s} a_i f_{n_i}\Big\|_{L_p} \le C\Big\|\sum_{i=1}^{s} a_i r_i\Big\|_{L_p},$$

where $C > 0$ depends only on D. Note that the results of [135] hold for vector-valued Rademacher series as well. Lemma 9.2 plays a very important role in the proof of Theorem 9.6. It belongs to Kashin (see [85] or [86, Lemma 9.4.1]). Lemma 9.3 is proved in [18]. Proposition 9.1 is proved in [132]. Our proof follows the one in [86] (see relations (71)–(73) in Chap. 4 there).

Chapter 10

RADEMACHER MULTIPLICATOR SPACES

Let X be a symmetric space of functions on $[0,1]$. Recall that $\mathcal{R}(X)$ is a closed subspace consisting of all functions $g:\ [0,1] \to \mathbb{R}$, $g \in X$ defined as follows: $g(t) = \sum\limits_{k=1}^{\infty} c_k r_k(t)$, where $r_k(t)$ are Rademacher functions.

Definition 10.1. The *Rademacher multiplicator space of a symmetric space* X is the set $\Lambda(\mathcal{R}, X)$ of all measurable functions f from $[0,1]$ to $\mathbb{R}$ such that $f \cdot g \in X$ for any $g \in \mathcal{R}(X)$.

Proposition 10.1. *For any symmetric space* X *on* $[0,1]$ *the set* $\Lambda(\mathcal{R}, X)$ *is a Banach ideal space with respect to the norm*

$$\|f\|_\Lambda = \sup\left\{\|f \cdot g\|_X :\ g \in \mathcal{R}(X), \|g\|_X \le 1\right\}.$$

Proof. First of all, applying the Banach–Steinhaus uniform boundedness principle, one can easily check that $\|f\|_\Lambda < \infty$ for any function $f \in \Lambda(\mathcal{R}, X)$. Since all properties of the norm for this functional and its monotonicity are checked, it remains to show that the space $\Lambda(\mathcal{R}, X)$ is complete.

Let the sequence $\{f_n\} \subset \Lambda(\mathcal{R}, X)$ be a Cauchy sequence in $\Lambda(\mathcal{R}, X)$. From the definition of the norm in $\Lambda(\mathcal{R}, X)$ it follows that for every $g \in \mathcal{R}(X)$ the sequence $f_n \cdot g$ is a Cauchy sequence in the space X. Since X is complete and the convergence implies the convergence in measure in every symmetric space, it follows that there exist a function f from X such that $\|f_n g - fg\|_X \to 0$ as $n \to \infty$ for every $g \in \mathcal{R}(X)$. Using the fact that $\{f_n\}$ is a Cauchy sequence in $\Lambda(\mathcal{R}, X)$, we fined for any $\varepsilon > 0$ a number N such that for all $m \ge n \ge N$ and $g \in \mathcal{R}(X)$, $\|g\|_X \le 1$, the inequality $\|f_n g - f_m g\|_X \le \varepsilon$ holds. Passing to the limit as $m \to \infty$, we obtain that $\|f_n g - fg\|_X \le \varepsilon$ if $g \in \mathcal{R}(X)$, $\|g\|_X \le 1$, which is equivalent to the inequality $\|f_n - f\|_{\Lambda(\mathcal{R},X)} \le \varepsilon$. Therefore, $f \in \Lambda(\mathcal{R}, X)$ and $f_n \to f$ in $\Lambda(\mathcal{R}, X)$. ☐

The term $\Lambda(\mathcal{R}, X)$ is explained by the following fact: We can treat this set as a space of multiplicators in X boundedly acting from the subspace $\mathcal{R}(X)$ to the whole space. In this chapter, we consider questions related to the symmetry of the lattice $\Lambda(\mathcal{R}, X)$. First, we show that it is not symmetric for a "majority" of symmetric spaces, i.e., the belonging of a function $f(t)$ to $\Lambda(\mathcal{R}, X)$ does not imply that the nonincreasing rearrangement $f^*(t)$ of its modulus belongs to $\Lambda(\mathcal{R}, X)$. Due to this, we introduce the definition of a symmetric kernel $\operatorname{Sym}(E)$ of the Banach lattice of measurable functions E, which is the greatest symmetric space included to E. First of all, we are interested in the symmetric kernel $\operatorname{Sym}(X)$ of the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$. For maximal symmetric space, we obtain their description.

The situation changes if X is quite "close" to the space L_∞. Simple and easily checked necessary and sufficient conditions of the relation $\Lambda(\mathcal{R}, X) = L_\infty$ (with norm equivalence) are obtained. For spaces "far" from L_∞ (e.g., exponential Orlicz spaces), the multiplicator space $\Lambda(\mathcal{R}, X)$ is also symmetric though it does not coincide with L_∞.

First, recall that, by Lemma 3.2, we have

$$\|x\|_{L_{N_2}} \asymp \sup_{0<t\le 1} x^*(t)\ln^{-1/2}(e/t),$$

where L_{N_2} is the Orlicz space constructed with respect to the function $N_2(u) = e^{u^2} - 1$. Therefore, there exists $C > 0$ such that for all $x \in L_{N_2}$ and $0 < t \le 1$ the following inequality holds:

$$x^*(t) \le C\,\|x\|_{L_{N_2}} \cdot \ln^{1/2}(e/t). \tag{10.1}$$

If $X \supset G$, where G is the separable part of L_{N_2}, then $\mathcal{R}(X) = \mathcal{R}(L_{N_2}) = \ell_2$ by Theorem 3.2. Applying the last inequality to the Rademacher sums, we obtain that there exists $K > 0$ such that

$$\Big(\sum_{n=1}^{\infty} a_n r_n\Big)^*(t) \le K \cdot \|(a_n)\|_2 \cdot \ln^{1/2}(e/t) \ \ (0 < t \le 1). \tag{10.2}$$

Begin with the following very important lemma on the comparison of multiplicator norms of characteristic functions.

Lemma 10.1. *Let X be a symmetric space on $[0,1]$ with fundamental function φ_X, $X \supset G$. Then there exists a constant $c > 0$ such that for any $n = 2^m$ $(m \in \mathbb{N})$ we can find measurable sets B_n and D_n of the same measure $n2^{-n}$ such that*

$$c\frac{\|\chi_{B_n}\|_{\Lambda(\mathcal{R},X)}}{\|\chi_{D_n}\|_{\Lambda(\mathcal{R},X)}} \le \frac{1}{n^{1/2}}\left(1 + \frac{1}{\varphi_X(n2^{-n})}\int_0^{n/2^n} \frac{\varphi_X(t)}{t}\,dt\right) + \Big(\frac{\varphi_X(2^{-n})}{\varphi_X(n2^{-n})}\Big)^{1/2}. \tag{10.3}$$

Proof. First, $\mathcal{R}(X) = \ell_2$ by condition, i.e., the norms $\|\sum a_n r_n\|_X$ and $\|(a_n)\|_2$ are equivalent to each other. Therefore, the following relation holds:

$$\|f\|_\Lambda \asymp \sup\left\{\|fg\|_X :\ g = \sum_{k=1}^{\infty} c_k r_k \in X, \|(c_n)\|_2 \le 1\right\}. \tag{10.4}$$

Let J be an arbitrary subset of the set $\{1, 2, \dots, 2^n\}$ consisting of n elements and $A = \cup_{j\in J}\Delta_j$ be the corresponding union of intervals $\Delta^j := \Delta_n^j = ((j-1)2^{-n}, j2^{-n})$ of rank n $(1 \le j \le 2^n)$. Note that $m(A) = n2^{-n}$. If χ_A is the characteristic function of the last set, then, due to (10.4), the following inequality holds:

$$\begin{aligned}\|\chi_A\|_\Lambda &\le C\sup\Big\{\Big\|\chi_A\sum_{i=1}^{\infty} b_i r_i\Big\|_X :\ \|(b_i)\|_2 \le 1\Big\}\\ &\le C\sup\Big\{\Big\|\chi_A\sum_{i=1}^{n} b_i r_i\Big\|_X :\ \|(b_i)\|_2 \le 1\Big\} + C\sup\Big\{\Big\|\chi_A\sum_{i=n+1}^{\infty} b_i r_i\Big\|_X :\ \|(b_i)\|_2 \le 1\Big\}.\end{aligned} \tag{10.5}$$

First, we estimate the second summand. Since for any $\lambda > 0$ we have

$$m\big(\{t : \chi_{[0,n2^{-n}]}(t)\Big|\sum_{i=n+1}^{\infty} b_i r_i(t)\Big| > \lambda\}\big) = \frac{n}{2^n}\, m\big(\{t : \Big|\sum_{i=n+1}^{\infty} b_i r_i(t)\Big| > \lambda\}\big),$$

it follows that

$$\Big(\chi_{[0,n2^{-n}]}\sum_{n+1}^{\infty} b_i r_i\Big)^*(t) = \Big(\sum_{n+1}^{\infty} b_i r_i\Big)^*(2^n t/n), \quad 0 \le t \le n/2^n.$$

Note that the functions

$$\chi_A(t)\cdot\sum_{i=n+1}^{\infty} b_i r_i(t) \quad \text{and} \quad \chi_{[0,n2^{-n}]}(t)\cdot\sum_{i=n+1}^{\infty} b_i r_i(t)$$

are equimeasurable. Then, due to $\|x\| \le \|x\|_{\Lambda(\varphi_X)}$ (see [92, Theorem 2.5.5]) and inequality (10.2), we see that

$$\Big\|\chi_A \sum_{i=n+1}^{\infty} b_i r_i\Big\|_X = \Big\|\chi_{[0,n2^{-n}]} \sum_{i=n+1}^{\infty} b_i r_i\Big\|_X \le \Big\|\chi_{[0,n2^{-n}]} \sum_{i=n+1}^{\infty} b_i r_i\Big\|_{\Lambda(\varphi_X)}$$

$$= \int_0^{n/2^n} \Big(\sum_{i=n+1}^{\infty} b_i r_i\Big)^* \Big(\frac{2^n t}{n}\Big) \varphi_X{}'(t)\,dt \le K\,\|(b_i)\|_2 \int_0^{n/2^n} \ln^{1/2}\Big(\frac{ne}{2^n t}\Big)\varphi_X{}'(t)\,dt.$$

Integrating by parts, we obtain that the last integral equals

$$\varphi_X\Big(\frac{n}{2^n}\Big) - \lim_{t\to 0} \ln^{1/2}\Big(\frac{ne}{2^n t}\Big)\varphi_X(t) + \frac{1}{2}\int_0^{n/2^n} \frac{\varphi_X(t)}{t\ln^{1/2}(ne/2^n t)}\,dt \le \varphi_X\Big(\frac{n}{2^n}\Big) + \frac{1}{2}\int_0^{n/2^n} \frac{\varphi_X(t)}{t}\,dt.$$

This yields the inequality

$$\Big\|\chi_A \sum_{i=n+1}^{\infty} b_i r_i\Big\|_X \le K\,\|(b_i)\|_2 \Big\{\varphi_X\Big(\frac{n}{2^n}\Big) + \frac{1}{2}\int_0^{n/2^n} \frac{\varphi_X(t)}{t}\,dt\Big\}. \tag{10.6}$$

Consider the first summand in (10.5). Let a_{ij} be the value of the function r_i ($i = 1, 2, \ldots, n$) on the dyadic interval Δ^j ($1 \le j \le 2^n$). Since $n = 2^m$, it follows that we can choose the set $J_1 \subset \{1, \ldots, 2^n\}$, $\operatorname{card} J_1 = n$, such that the matrix $n^{-1/2}(a_{ij})_{j\in J_1}$ is orthogonal. Then, for $B_n = \cup_{j\in J_1}\Delta_j$, we obtain that

$$\sup\Big\{\Big\|\chi_{B_n}\sum_{i=1}^{n} b_i r_i\Big\|_X : \|(b_i)_1^n\|_2 \le 1\Big\} = \sup\Big\{\Big\|\sum_{j\in J_1}\Big(\sum_{i=1}^{n} b_i r_i\Big)\chi_{\Delta_j}\Big\|_X : \|(b_i)_1^n\|_2 \le 1\Big\}$$

$$= \sup\Big\{\Big\|\sum_{j\in J_1} c_j\chi_{\Delta_j}\Big\|_X : \|(c_j)_{j\in J_1}\|_2 \le n^{1/2}\Big\} = n^{1/2}\sup\Big\{\Big\|\sum_{j\in J_1} c_j\chi_{\Delta_j}\Big\|_X : \|(c_j)_{j\in J_1}\|_2 \le 1\Big\}.$$

Therefore, due to (10.5) and (10.6), the following inequality holds:

$$\|\chi_{B_n}\|_\Lambda \le K\Big\{\varphi_X\Big(\frac{n}{2^n}\Big) + \frac{1}{2}\int_0^{n/2^n}\frac{\varphi_X(t)}{t}\,dt\Big\} + n^{1/2}\sup\Big\{\Big\|\sum_{j\in J_1} c_j\chi_{\Delta_j}\Big\|_X : \|(c_j)_{j\in J_1}\|_2 \le 1\Big\}. \tag{10.7}$$

On the other hand, take the set $J_2 \subset \{1, \ldots, 2^n\}$, $\operatorname{card} J_2 = n$, such that all elements of every column of the matrix $(a_{ij})_{j\in J_2}$, except one, equal 1. Then for every $j \in J_2$ we have $\sum_{i=1}^{n} a_{ij} = n - 2$. Therefore, since $D_n = \cup_{j\in J_2}\Delta_j$, we have

$$\|\chi_{D_n}\|_\Lambda \ge c_1\Big\|\sum_{j\in J_2}\Big(\sum_{i=1}^{n}\frac{a_{ij}}{n^{1/2}}\Big)\chi_{\Delta_j}\Big\|_X = c_1(n^{1/2} - 2n^{-1/2})\cdot\varphi_X(n2^{-n}). \tag{10.8}$$

From (10.7) and (10.8), it follows that

$$\begin{aligned} c\frac{\|\chi_{B_n}\|_{\Lambda(\mathcal{R},X)}}{\|\chi_{D_n}\|_{\Lambda(\mathcal{R},X)}} &\le \frac{1}{n^{1/2}}\Big(1 + \frac{1}{\varphi_X(n2^{-n})}\int_0^{n/2^n}\frac{\varphi_X(t)}{t}\,dt\Big) \\ &+ \varphi_X(n2^{-n})^{-1}\sup\Big\{\Big\|\sum_{j\in J_1} c_j\chi_{\Delta_j}\Big\|_X : \|(c_j)_{j\in J_1}\|_2 \le 1\Big\}, \end{aligned} \tag{10.9}$$

where $C > 0$ does not depend on n. Estimate the second summand from the right-hand side of the last relation. Let X_n be an n-dimensional subspace of X, generated by the functions χ_{Δ_j} ($j \in J_1$).

Consider the linear operator $T : \mathbb{R}^n \to X_n$ defined by the relation $T(c_j) = \sum\limits_{j\in J_1} c_j\chi_{\Delta_j}$. One can easily see that $\|T\|_{\ell_\infty^n\to X_n} = \varphi_X(n2^{-n})$ and $\|T\|_{\ell_1^n\to X_n} = \varphi_X(2^{-n})$. Since the adjoint operator $T^* : X_n^* \to \mathbb{R}^n$ is defined by the relation $T^*x^* = (x^*(\chi_{\Delta_j}))$ and

$$\sum_{i=1}^n x_i^2 \le \max_{i=1,\dots,n} |x_i| \cdot \sum_{i=1}^n |x_i|,$$

it follows that

$$\|T^*\|_{X_n^*\to\ell_2^n} \le \|T^*\|^{1/2}_{X_n^*\to\ell_1^n} \cdot \|T^*\|^{1/2}_{X_n^*\to\ell_\infty^n}.$$

Therefore, $\|T\|_{\ell_2^n\to X_n} \le \|T\|^{1/2}_{\ell_\infty^n\to X_n} \cdot \|T\|^{1/2}_{\ell_1^n\to X_n}$, whence we see that

$$\varphi_X(n2^{-n})^{-1} \sup\Big\{\Big\|\sum_{j\in J_1} c_j\chi_{\Delta_j}\Big\|_X : \|(c_j)_{j\in J_1}\|_2 \le 1\Big\}$$

$$= \|T\|^{-1}_{\ell_\infty^n\to X_n} \cdot \|T\|_{\ell_2^n\to X_n} \le \|T\|^{-1/2}_{\ell_\infty^n\to X_n} \cdot \|T\|^{1/2}_{\ell_1^n\to X_n} = \left(\frac{\varphi_X(2^{-n})}{\varphi_X(n2^{-n})}\right)^{1/2}.$$

This and (10.9) imply the statement of the lemma. □

Now it is easy to show that for a symmetric space X quite "far" from L_∞, the Banach lattice $\Lambda(\mathcal{R}, X)$ is not symmetric.

Theorem 10.1. *If the lower dilation index of the fundamental function of a symmetric space X is positive, i.e., $\gamma_{\varphi_X} > 0$, then $\Lambda(\mathcal{R}, X)$ is not a symmetric space. In particular, this is valid if the lower Boyd index α is positive.*

Proof. If $\gamma_{\varphi_X} > 0$, then for every $\alpha \in (0, \gamma_{\varphi_X})$ there exists a constant $C = C(\alpha) > 0$ such that for all $s > 0$ and $t \le 1$ the following inequality holds:

$$\frac{\varphi_X(st)}{\varphi_X(s)} \le C\,t^\alpha. \tag{10.10}$$

This implies the inequality $\varphi_X(t) \le C\varphi_X(1)t^\alpha$; therefore, $\Lambda(t^\alpha) \subset \Lambda(\varphi_X)$. One can easily check that $L_p \subset \Lambda(t^\alpha)$ if $p > 1/\alpha$ (see also [92, Theorem 2.5.6]). Since $L_{N_2} \subset L_p$ for all $p < \infty$ and $\Lambda(\varphi_X) \subset X$, it follows that $L_{N_2} \subset X$. Therefore, the condition of the previous lemma holds and for every $n = 2^m$ ($m \in \mathbb{N}$) there exist measurable sets B_n and D_n of measure $n2^{-n}$ such that (10.3) holds.

It follows from (10.10) that $\varphi_X(2^{-n})/\varphi_X(n2^{-n}) \le C\,n^{-\alpha}$. Therefore, the second summand from the right part of (10.3) converges to zero as $n \to \infty$. Regarding the first summand, due to the condition $\gamma_{\varphi_X} > 0$, we can apply [92, Corollary 2, p. 80]; therefore, there exists $C > 0$ such that

$$\int_0^{n/2^n} \frac{\varphi_X(t)}{t}\,dt \le C\varphi_X(n/2^n).$$

Therefore, this summand from the right part of (10.3) is majorized by $Cn^{-1/2}$. Therefore, the fraction $\|\chi_{B_n}\|_\Lambda \|\chi_{D_n}\|_\Lambda{}^{-1}$ tends to zero as $n \to \infty$, and this implies that the space $\Lambda(\mathcal{R}, X)$ is not symmetric.

The last statement of the theorem follows from the inequality $\gamma_{\varphi_X} \ge \alpha_X$. □

Due to the last theorem, the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ is not symmetric for the "majority" of symmetric spaces X (e.g., for L_p, $1 \le p < \infty$). Therefore, the problem of the description of the largest symmetric space embedded into $\Lambda(\mathcal{R}, X)$ looks natural and interesting.

In the sequel, we will use the following notation. If $x(t)$ and $y(t)$ are measurable functions on $[0, 1]$, then we write $y \simeq x$, if the absolute values of these functions are equimeasurable, i.e., for any $\tau > 0$ we have

$$m\{t \in [0, 1] : |x(t)| > \tau\} = m\{t \in [0, 1] : |y(t)| > \tau\}.$$

Definition 10.2. Let E be a Banach ideal space of measurable functions on [0,1]. The *Symmetric kernel* E is a set

$$\mathrm{Sym}\,(E) = \{x \in E: \text{ if } y \simeq x, \quad \text{then} \quad y \in E\}.$$

For simplicity we write $\mathrm{Sym}\,(\mathcal{R}, X)$ instead of $\mathrm{Sym}\,(\Lambda(\mathcal{R}, X))$.

Proposition 10.2. *Let X be a symmetric space on $[0,1]$. The symmetric kernel $\mathrm{Sym}\,(\mathcal{R}, X)$ of the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ with respect to the norm*

$$\|x\|_{\mathrm{Sym}\,(\mathcal{R},X)} := \sup_{z:\, z \simeq x} \|z\|_{\Lambda(\mathcal{R},X)} \tag{10.11}$$

is the largest symmetric space, continuously embedded into the space $\Lambda(\mathcal{R}, X)$.

Proof. Assume that $x \in \mathrm{Sym}\,(\mathcal{R}, X)$. For arbitrary $z \simeq x$, we consider the operator T_z defined by the relation $T_z y(t) := z(t)y(t)$, where $y \in \mathcal{R}(X)$. Since $z \in \Lambda(\mathcal{R}, X)$, it follows that T_z boundedly acts from $\mathcal{R}(X)$ to X. For any $y \in \mathcal{R}(X)$ and $\varepsilon > 0$ there exist a measure-preserving transformation $\omega\colon [0,1] \to [0,1]$ such that $\|\, |y| - y^*(\omega)\|_\infty \le \varepsilon$ (see [92, Lemma 2.2.1, p. 84]). If $x \in \mathrm{Sym}\,(\mathcal{R}, X)$ and $y \in \mathcal{R}(X)$, then $x^*(\omega)y \in X$ and

$$\|T_{z^*}y^*\|_X = \|z^*(\omega)y^*(\omega)\|_X \le \|z^*(\omega)y\|_X + \|z^*(\omega)(|y| - y^*(\omega))\|_X \le \|x^*(\omega)y\|_X + \varepsilon\|x\|_X.$$

Therefore, due to the inequality

$$(T_z y)^*(2t) \le T_{z^*}y^*(t), \quad 0 < t < 1/2,$$

(see [92, property 10°, p. 93]) and the boundedness of the dilation operator, we obtain the inequality

$$\sup_{z \simeq x} \|T_z y\|_X \le 2\|T_{z^*}y^*\|_X \le 2(\|x^*(\omega)y\|_X + \varepsilon\|x\|_X) < \infty.$$

Finally, the family of operators $\{T_z : z \simeq x\}$ is pointwise bounded with respect to the norm; therefore, by the Banach–Steinhaus theorem, it is uniformly bounded. This means that the right-hand side of (10.11) is finite for any $x \in \mathrm{Sym}\,(\mathcal{R}, X)$. Let us check that it fulfills the triangle inequality.

Let x_1 and x_2 belong to the set $\mathrm{Sym}\,(\mathcal{R}, X)$ and $z \simeq x_1 + x_2$. Due to [92, Lemma 2.2.1, p. 84], for any $\varepsilon > 0$ there exists a measure-preserving transformation $\tau\colon [0,1] \to [0,1]$ such that $\||z| - |x_1(\tau) + x_2(\tau)|\|_\infty \le \varepsilon$. Therefore, for arbitrary $y \in \mathcal{R}(X)$, $\|y\|_X \le 1$, the following inequality holds:

$$\begin{aligned} \|zy\|_X &\le \|(|z| - |x_1(\tau) + x_2(\tau)|)y\|_X + \|(x_1(\tau) + x_2(\tau))y\|_X \\ &\le \varepsilon + \|x_1(\tau)y\|_X + \|x_2(\tau))y\|_X \le \varepsilon + \|x_1\|_{\mathrm{Sym}\,(\mathcal{R},X)} + \|x_2\|_{\mathrm{Sym}\,(\mathcal{R},X)}. \end{aligned}$$

Therefore, $x_1 + x_2 \in \mathrm{Sym}\,(\mathcal{R}, X)$ and

$$\|x_1 + x_2\|_{\mathrm{Sym}\,(\mathcal{R},X)} \le \|(x_1\|_{\mathrm{Sym}\,(\mathcal{R},X)} + \|x_2\|_{\mathrm{Sym}\,(\mathcal{R},X)} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that the triangle inequality is proved. Other properties of the norm obviously hold.

The fact that $\mathrm{Sym}\,(\mathcal{R}, X)$ is the largest symmetric space continuously embedded into $\Lambda(\mathcal{R}, X)$ follows from the definition of the symmetric kernel and its norm. Therefore, it remains to prove the completeness. If $\{x_n\} \subset \mathrm{Sym}\,(\mathcal{R}, X)$ is a a Cauchy sequence in $\mathrm{Sym}\,(\mathcal{R}, X)$, then it is a Cauchy sequence in $\Lambda(\mathcal{R}, X)$; therefore, there exists $x \in \Lambda(\mathcal{R}, X)$ such that $\|x_n - x\|_{\Lambda(\mathcal{R},X)} \to 0$. It is clear that for a measure-preserving transformation $\lambda\colon [0,1] \to [0,1]$, the sequence $\{x_n(\lambda)\} \subset \mathrm{Sym}\,(\mathcal{R}, X)$ is a Cauchy sequence in this space. Therefore, there exists a function $x_\lambda \in \Lambda(\mathcal{R}, X)$ such that $\|x_n(\lambda) - x_\lambda\|_{\Lambda(\mathcal{R},X)} \to 0$. Due to the convergence in measure in any Banach ideal space, it follows that $x_\lambda(t) = x(\lambda(t))$ for almost every $t \in [0,1]$. Using the fact that $\{x_n\}$ is a Cauchy sequence in $\mathrm{Sym}\,(\mathcal{R}, X)$ for any $\varepsilon > 0$, we find a number N such that for all $m \ge n \ge N$ and any measure-preserving transformation $\lambda\colon [0,1] \to [0,1]$ the following inequality holds:

$$\|x_n(\lambda) - x_m(\lambda)\|_\Lambda \le \varepsilon.$$

Passing to the limit as $m \to \infty$, we obtain the inequality

$$\|x_n(\lambda) - x(\lambda)\|_\Lambda \le \varepsilon. \tag{10.12}$$

Let $z \simeq x_n - x$ for a fixed $n \ge N$. As above, we can find a measure-preserving transformation $\lambda_0\colon [0,1] \to [0,1]$ such that $\||z| - |x_n(\lambda_0) - x(\lambda_0)|\|_\infty \le \varepsilon$. Then, similarly to the proof of the triangle inequality for arbitrary $y \in \mathcal{R}(X)$, $\|y\|_X \le 1$, due to (10.12), we obtain the inequality

$$\|zy\|_X \le \varepsilon + \|(x_n(\lambda_0) - x(\lambda_0))y\|_X \le 2\varepsilon,$$

i.e., $\|x_n - x\|_{\mathrm{Sym}\,(\mathcal{R},X)} \le 2\varepsilon$. This implies that $x \in \mathrm{Sym}\,(\mathcal{R}, X)$ and $x_n \to x$ in $\mathrm{Sym}\,(\mathcal{R}, X)$. □

From Proposition 10.2 and the inequality $(xy)^*(2t) \le x^*(t)y^*(t)$ $(0 < t \le 1/2)$ (see [92, property 10°, p. 93]), the following description of the symmetric kernel $\mathrm{Sym}\,(\mathcal{R}, X)$ of the multiplicator space $\Lambda(\mathcal{R}, X)$ follows.

Corollary 10.1. *For any symmetric space X, a space* $\mathrm{Sym}\,(\mathcal{R}, X)$ *consists of all $x = x(t)$ fulfilling the following condition: if $\sum\limits_{n=1}^{\infty} a_n r_n \in \mathcal{R}(X)$, then $x^*(\sum\limits_{n=1}^{\infty} a_n r_n)^* \in X$. Moreover,*

$$\|f\|_{\mathrm{Sym}\,(\mathcal{R},X)} \asymp \sup_{\|\sum a_n r_n\|_X \le 1} \Big\| f^* \Big(\sum_{n=1}^{\infty} a_n r_n \Big)^* \Big\|_X.$$

The following definition is needed for the description of symmetric kernels of Rademacher multiplicator spaces.

Definition 10.3. If X is a symmetric space on $[0,1]$ such that $L_{N_2} \subset X$, then the set $X_{\ln^{1/2}}$ consists of all measurable on $[0,1]$ functions $x = x(t)$ such that

$$\|x\|_{\ln^{1/2}} := \|x^*(t) \ln^{1/2}(e/t)\|_X < \infty.$$

Since due to Lemma 3.2, the condition $L_{N_2} \subset X$ is equivalent to $\ln^{1/2}(e/t) \in X$, it follows that the set $X_{\ln^{1/2}}$ contains the space L_∞.

The functional $\|\cdot\|_{\ln^{1/2}}$ is a quasinorm (the triangle inequality holds for it with some constant). At the same time, if $X \in \mathcal{I}\,(L_1, L_\infty)$, then it is equivalent to the norm; therefore, $X_{\ln^{1/2}}$ becomes a symmetric space Indeed, (L_1, L_∞) is a $\mathcal{K}$-monotone couple, hence, by the Brudnyi–Kruglyak theorem (see Sec. 0.5 or [50, Theorem 15.1] for details), there exists a parameter F of the $\mathcal{K}$-method such that $X = (L_1, L_\infty)^{\mathcal{K}}_F$. Since

$$\mathcal{K}(t, x; L_1, L_\infty) = \int_0^t x^*(s)\, ds$$

(see [40, Theorem 5.2.1]) and the function $\ln^{1/2}(e/t)$ decreases, we have the relation

$$\|x\|_{\ln^{1/2}} \asymp \Big\| \Big(\int_0^{2^k} x^*(t) \ln^{1/2}(e/t)\, dt \Big)_k \Big\|_F.$$

Note that the right-hand side of the last expression is subadditive with respect to the function $x(t)$ (see [92, property 8°, p. 89]).

Theorem 10.2. *Let X be a symmetric space on* $[0,1]$ *and $L_{N_2} \subset X$. Then*

$$X_{\ln^{1/2}} \subset \mathrm{Sym}\,(\mathcal{R}, X) \subset (X'')_{\ln^{1/2}}.$$

Proof. Since $L_{N_2} \subset X$, it follows from Theorem 3.2 that $\mathcal{R}(X) = \ell_2$. First, assume that $x^*(t)\ln^{1/2}(e/t) \in X$. If $(a_n) \in \ell_2$, then, applying (10.2), we obtain that for every $0 < t \leq 1$, the following inequality holds:

$$x^*(t)\Big(\sum_{n=1}^{\infty} a_n r_n\Big)^*(t) \leq K\|(a_n)\|_2 \; x^*(t)\ln^{1/2}(e/t).$$

Therefore, $x^*\big(\sum\limits_{n=1}^{\infty} a_n r_n\big)^* \in X$. Hence, $x \in \mathrm{Sym}\,(\mathcal{R}, X)$ by Corollary 10.1 and

$$\|x\|_{\mathrm{Sym}\,(\mathcal{R},X)} \leq C_1 \, \|x\|_{X_{\ln^{1/2}}}.$$

Now, let $x \in \mathrm{Sym}\,(\mathcal{R}, X)$. Introduce $v_n = n^{-1/2} \sum\limits_{i=1}^{n} r_i$. Since $\mathcal{R}(X) = \ell_2$, it follows that

$$\sup_{n=1,2,..} \|v_n\|_X < \infty.$$

Therefore, by Corollary 10.1, the following inequality holds:

$$\|x^* v_n^*\|_X \leq C_2 \, \|x\|_{\mathrm{Sym}\,(\mathcal{R},X)}. \tag{10.13}$$

Moreover, as in the proof of Theorem 3.2, we obtain the inequality

$$\lim_{n\to\infty} v_n^*(t) \geq c\ln^{1/2}(e/t) \;\; (0 < t \leq 1).$$

Therefore, there exists $C_3 > 0$ such that

$$x^*(t)\ln^{1/2}(e/t) \leq C_3 \, x^*(t) \lim_{n\to\infty} v_n^*(t) \;\; (0 < t \leq 1).$$

Therefore, since X'' is maximal, it follows from (10.13) that $x^*(t)\ln^{1/2}(e/t) \in X''$ and

$$\|x^*(t)\ln^{1/2}(e/t)\|_{X''} \leq C_3 \, \liminf_{n\to\infty} \|x^* v_n^*\|_X \leq C_2 \, C_3 \, \|x\|_{\mathrm{Sym}\,(\mathcal{R},X)}.$$

□

Remark 10.1. The same proof shows that the embedding

$$\mathrm{Sym}\,(\mathcal{R}, X) \subset (X'')_{\ln^{1/2}}$$

holds even under the following weaker condition: $G \subset X$, i.e., if $L_{N_2} \subset X''$.

Remark 10.2. Since $\mathrm{Sym}\,(\mathcal{R}, L_1) = (L_1)_{\ln^{1/2}} = L \log^{1/2} L$, it follows from Theorem 10.2 that

$$L_\infty \subset \mathrm{Sym}\,(\mathcal{R}, X) \subset L \log^{1/2} L$$

for every symmetric space X

If X is maximal, then $X = X''$. Therefore, from the theorem 10.2 we obtain

Corollary 10.2. *Let X be a maximal symmetric space such that $L_{N_2} \subset X$. Then*

$$\mathrm{Sym}\,(\mathcal{R}, X) = X_{\ln^{1/2}} \;\; (\textit{with equivalence of norms}).$$

Corollary 10.2 allows us to find $\mathrm{Sym}\,(\mathcal{R}, X)$ for different classes of symmetric spaces

Example 10.1. For $X = L_p$, $1 \leq p < \infty$ we have

$$\|x\|_{\mathrm{Sym}\,(\mathcal{R},X)} \asymp \Big(\int_0^1 \big(x^*(t)\ln^{1/2}(e/t)\big)^p \, dt\Big)^{1/p},$$

i.e., $\mathrm{Sym}\,(\mathcal{R}, L_p)$ is a Zygmund space $L_p(\log L)^{1/2}$ (see [39, Def. 4.6.11]).

Example 10.2. Consider a more general case of Lorentz–Zygmund spaces (see [39, Def. 4.6.13]). The space $X = L_{p,q}(\log L)^\alpha$ consist of all measurable on $[0,1]$ functions $x(t)$ such that

$$\|x\|_{p,q,\alpha} = \Big(\int_0^1 \big(t^{1/p} \ln^\alpha(e/t) x^*(t)\big)^q \,\frac{dt}{t}\Big)^{1/q} < \infty$$

if either $1 < p < \infty$, $1 \le q < \infty$, and $\alpha \in \mathbb{R}$, or $p = q = 1$ and $\alpha \ge 0$, and

$$\|x\|_{p,\infty,\alpha} = \sup_{0<t\le 1} t^{1/p} \ln^\alpha(e/t) x^*(t) < \infty$$

if $1 < p < \infty$ and $\alpha \in \mathbb{R}$. In the first case, we have

$$\|x\|_{\mathrm{Sym}\,(\mathcal{R},X)} \asymp \Big(\int_0^1 \big(t^{1/p} \ln^{1/2+\alpha}(e/t) x^*(t)\big)^q \,\frac{dt}{t}\Big)^{1/q}.$$

In the second case, we have

$$\|x\|_{\mathrm{Sym}\,(\mathcal{R},X)} \asymp \sup_{0<t\le 1} t^{1/p} \ln^{1/2+\alpha}(e/t) x^*(t).$$

Therefore, $\mathrm{Sym}\,(\mathcal{R}, L_{p,q}(\log L)^\alpha)$ is a Lorentz–Zygmund space $L_{p,q}(\log L)^{1/2+\alpha}$ in both cases. In particular, if $\alpha = 0$, then we obtain the Marcinkiewicz space $L_{p,\infty}$ generated by the power function $t^{1-1/p}$; then $\mathrm{Sym}\,(\mathcal{R}, L_{p,\infty}) = L_{p,\infty}(\log L)^{1/2}$.

Example 10.3. Let $X = \Lambda_p(\varphi)$ be a Lorentz space (φ is an increasing concave function such that $\varphi(0) = 0$ and $1 \le p < \infty$). The embedding $L_{N_2} \subset \Lambda_p(\varphi)$ holds if and only if $\int_0^1 \ln^{p/2}(e/t)\, d\varphi(t) < \infty$. Then

$$\|x\|_{\mathrm{Sym}\,(\mathcal{R},X)} \asymp \Big(\int_0^1 \big(x^*(t)\big)^p \ln^{p/2}(e/t)\ d\varphi(t)\Big)^{1/p}.$$

The function $\varphi'(t) \ln^{p/2}(e/t)$ is non-negative, integrable, and decreasing. Therefore, $\mathrm{Sym}\,(\mathcal{R}, \Lambda_p(\varphi))$ is a Lorentz space $\Lambda_p(\psi_p)$, where

$$\psi_p(s) = \int_0^s \varphi'(t) \ln^{p/2}(e/t)\, dt.$$

In particular, for $p = 1$, we obtain the relation $\mathrm{Sym}\,(\mathcal{R}, \Lambda(\varphi)) = \Lambda(\psi_1)$.

Example 10.2 shows that for any $1 < p < \infty$ there exists a symmetric space X such that $\mathrm{Sym}\,(\mathcal{R}, X) = L_p$. Indeed, for X we can take the Zygmund space $L_p(\log L)^{-1/2}$. Consider the more general question about conditions under which symmetric spaces can be obtained as symmetric kernels of Rademacher multiplicator spaces of a symmetric space. Remark 10.2 gives us a necessary condition: $Z \subset L \log^{1/2} L$. Before formulating the inverse statement, we prove the following lemma.

Lemma 10.2. *Let $\alpha > 0$ and $\gamma \in \mathbb{R}$. Then, there exist constants depending on α and γ, but not depending on the function $x(t)$, measurable on $[0,1]$, such that the following relation holds*:

$$\int_0^1 \big(x^*(\cdot) \ln^\gamma(e/\cdot)\big)^*(t) \ln^\alpha(e/t)\, dt \asymp \int_0^1 x^*(t) \ln^{\gamma+\alpha}(e/t)\, dt.$$

Proof. It suffices to consider the case where $\gamma < 0$. Since $\alpha > 0$, it follows that $\ln^\alpha(e/t)$ decreases. Therefore, the following inequality holds:

$$\int_0^1 x^*(t)\ln^{\gamma+\alpha}(e/t)\,dt \le \int_0^1 \big(x^*(\cdot)\ln^\gamma(e/\cdot)\big)^*(t)\ln^\alpha(e/t)\,dt.$$

Prove that there exists a constant such that the inverse inequality holds. First, let $x(t) = \chi_{(0,a]}(t)$, $0 < a < 1$. Since $\int_0^t \ln^\beta(e/s)\,ds \asymp t\,\ln^\beta(e/t)$ for any $\beta \in \mathbb{R}$, it follows that

$$\int_0^a \big(\ln^\gamma(e/\cdot)\big)^*(t)\ln^\alpha(e/t)\,dt = \int_0^a \ln^\gamma\Big(\frac{e}{a-t}\Big)\ln^\alpha(e/t)\,dt$$
$$\le \ln^\gamma(e/a)\int_0^a \ln^\alpha(e/t)\,dt \asymp \int_0^a \ln^{\gamma+\alpha}(e/t)\,dt.$$

Therefore, we proved the necessary inequality for characteristic functions.

If the rearrangement $x^*(t)$ is finite-valued, then we can represent it as follows:

$$x^*(t) = \sum_{k=1}^N b_k\chi_{(0,a_k]}(t)\ \ (b_k > 0,\ a_1 > a_2 > \ldots > a_N \ge 0). \tag{10.14}$$

Then, due to properties of non-increasing rearrangements (see [92, property 14°, p. 95]), we have

$$\int_0^1 \big(x^*(\cdot)\ln^\gamma(e/\cdot)\big)^*(t)\ln^\alpha(e/t)\,dt = \sup_\omega \int_0^1 x^*(\omega(t))\ln^\gamma(e/\omega(t))\ln^\alpha(e/t)\,dt,$$

where the least upper bound is taken over all measure-preserving transformations $\omega : [0,1] \to [0,1]$. Therefore, using the above case, we obtain the inequality

$$\int_0^1 \big(x^*(\cdot)\ln^\gamma(e/\cdot)\big)^*(t)\ln^\alpha(e/t)\,dt \le \sum_{k=1}^N b_k \sup_\omega \int_0^1 \chi_{(0,a_k]}(\omega(t))\ln^\gamma(e/\omega(t))\ln^\alpha(e/t)\,dt$$
$$\le \sum_{k=1}^N b_k \int_0^{a_k} \big(\ln^\gamma(e/\cdot)\big)^*(t)\ln^\alpha(e/t)\,dt \le C\sum_{k=1}^N b_k\int_0^{a_k}\ln^{\gamma+\alpha}(e/t)\,dt = C\int_0^1 x^*(t)\ln^{\gamma+\alpha}(e/t)\,dt.$$

It is clear that an arbitrary rearrangement $x^*(t)$ is the limit of increasing sequence of functions $x_n^*(t)$ of form (10.14). Thus, as we have already proved

$$\int_0^1 \big(x_n^*(\cdot)\ln^\gamma(e/\cdot)\big)^*(t)\ln^\alpha(e/t)\,dt \le C\int_0^1 x_n^*(t)\ln^{\gamma+\alpha}(e/t)\,dt \le C\int_0^1 x^*(t)\ln^{\gamma+\alpha}(e/t)\,dt.$$

Finally, using Lemma 3.1 and passing to the limit on the left-hand side as $n \to \infty$, we come to the required inequality. ◻

Theorem 10.3. *Let Z be a maximal symmetric space such that $Z \in \mathcal{I}\,(L\log^{1/2}L, L_\infty)$. Then there exists a symmetric space X such that* $\mathrm{Sym}\,(\mathcal{R}, X) = Z$.

Proof. For a given Z, we define a space $Z_{\ln^{-1/2}}$ as the set of all functions $x = x(t)$ measurable on $[0,1]$ and such that

$$\|x\|_{\ln^{-1/2}} = \|x^*(t)\ln^{-1/2}(e/t)\|_Z < \infty.$$

Show that the functional $\|\cdot\|_{\ln^{-1/2}}$ is equivalent to the norm of some symmetric space First, calculate the $\mathcal{K}$-functional for the couple $(L\log^{1/2}L, L_\infty)$. The spaces L_∞ and $L\log^{1/2}L$ are Lorentz spaces. Indeed, $L_\infty = \Lambda(\varphi_0)$, where $\varphi_0(t) = 1$, and one can easily check that $L\log^{1/2}L = \Lambda(\varphi_1)$, where $\varphi_1(t) = t\ln^{1/2}(e/t)$. Since the function φ_0/φ_1 decreases, it follows from [92, Theorem 2.5.9] that

$$\mathcal{K}(t,x;L\log^{1/2}L,L_\infty) = \int_0^1 x^*(s)\,d(\min\{\varphi_1(s),t\varphi_0(s)\}) = \int_0^{s(t)} x^*(s)\,d(s\ln^{1/2}(e/s))$$

for all $t\in(0,1]$, where $s=s(t)$ is such that $s\ln^{1/2}(e/s)=t$, whence $s(t)\asymp t\ln^{-1/2}(e/t)$. Due to the fact that $(s\ln^{1/2}(e/s))'\asymp\ln^{1/2}(e/s)$, we obtain the relation

$$\mathcal{K}(t,x;L\log^{1/2}L,L_\infty) \asymp \int_0^{t\ln^{-1/2}(e/t)} x^*(s)\ln^{1/2}(e/s)\,ds \quad (0<t\le 1). \tag{10.15}$$

Since $Z\in\mathcal{I}(L\log^{1/2}L,L_\infty)$ and this couple is $\mathcal{K}$-monotone (see Proposition 5.2), it follows from the Brudnyi–Kruglyak theorem that there exists a Banach ideal space E of two-sided sequences such that $(\min(1,2^k))_{-\infty}^{\infty}\in E$ and

$$\|x\|_Z \asymp \|(\mathcal{K}(2^k,x;L\log^{1/2}L,L_\infty))_{k=-\infty}^{\infty}\|_E. \tag{10.16}$$

Denote by $\{e_k\}_{k=-\infty}^{\infty}$ the canonical basis in the sequence space E. Since $L_\infty\subset L\log^{1/2}L$ with constant 1, it follows that $\mathcal{K}(t,x;L\log^{1/2}L,L_\infty)=\|x\|_{L\log^{1/2}L}$ if $t\ge 1$. Moreover, due to the concavity of the $\mathcal{K}$-functional with respect to t (see [40, Lemma 3.1.1]), we have

$$\mathcal{K}(1,x;L\log^{1/2}L,L_\infty)\le 2\,\mathcal{K}(2^{-1},x;L\log^{1/2}L,L_\infty).$$

Therefore,

$$\Big\|\sum_{k=0}^{\infty}\mathcal{K}(2^k,x;L\log^{1/2}L,L_\infty)e_k\Big\|_E = \mathcal{K}(1,x;L\log^{1/2}L,L_\infty)\,\Big\|\sum_{k=0}^{\infty}e_k\Big\|_E$$
$$\le C\,\mathcal{K}(2^{-1},x;L\log^{1/2}L,L_\infty)\,\|e_{-1}\|_E \le C\,\Big\|\sum_{k=-1}^{-\infty}\mathcal{K}(2^k,x;L\log^{1/2}L,L_\infty)e_k\Big\|_E.$$

Therefore, from (10.16), we obtain the inequality

$$\|x\|_Z \asymp \Big\|\sum_{k=-1}^{-\infty}\mathcal{K}(2^k,x;L\log^{1/2}L,L_\infty)e_k\Big\|_E.$$

Therefore, due to (10.15) and Lemma 10.2, we obtain the inequality

$$\|x\|_{\ln^{-1/2}} \asymp \Big\|\sum_{k=-1}^{-\infty}\int_0^{2^k\ln^{-1/2}(e2^{-k})}\big(x^*(\cdot)\ln^{-1/2}(e/\cdot)\big)^*(t)\ln^{1/2}(e/t)\,dt\;e_k\Big\|_E$$
$$\asymp \Big\|\sum_{k=-1}^{-\infty}\int_0^{2^k\ln^{-1/2}(e2^{-k})} x^*(t)\,dt\;e_k\Big\|_E.$$

The last relation shows that the functional $\|\cdot\|_{\ln^{-1/2}}$ is equivalent to some symmetric norm [92, property 8°, p. 89]. Moreover, this implies that $Z_{\ln^{-1/2}}\in\mathcal{I}(L_1,L_\infty)$.

Finally, we check that the space $X := Z_{\ln^{-1/2}}$ is maximal. Indeed, if $0 \le x_n(t) \uparrow x(t)$ a.e. and $\sup_n \|x_n\|_{\ln^{-1/2}} < \infty$, then, by Lemma 3.1, we see that

$$x_n^*(t) \ln^{-1/2}(e/t) \uparrow x^*(t) \ln^{-1/2}(e/t) \ \text{ a.e.} \quad \text{and} \quad \sup_{n=1,2,\dots} \|x_n^* \ln^{-1/2}(e/\cdot)\|_Z < \infty.$$

Since Z is maximal, it follows that $x^* \ln^{-1/2}(e/\cdot) \in Z$, i.e., $x \in X$. Therefore, by Corollary 10.2, we obtain the relation

$$\|x\|_{\operatorname{Sym}(\mathcal{R},X)} \asymp \|x^* \ln^{1/2}(e/t)\|_X = \|\big(x^*(\cdot) \ln^{1/2}(e/\cdot)\big)^*(t) \ln^{-1/2}(e/t)\|_Z = \|x\|_Z.$$

□

Consider a symmetric kernel $\operatorname{Sym}(\mathcal{R}, X)$, assuming that X is a Marcinkiewicz space. We find conditions proving that $\operatorname{Sym}(\mathcal{R}, X)$ is a Marcinkiewicz space as well. The results obtained will be used to study the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$.

Theorem 10.4. *Let φ be a function quasiconcave on $[0,1]$ and such that the corresponding Marcinkiewicz space $M(\varphi)$ contains the space L_{N_2}. Introduce the following conditions*:

(i) *There exists a constant $C > 0$ such that*

$$\int_0^t \frac{\varphi(s)}{s} \ln^{-1}(e/s)\, ds \le C\, \varphi(t)$$

for every $0 \le t \le 1$;

(ii) $\operatorname{Sym}(\mathcal{R}, M(\varphi))$ *is a Marcinkiewicz space.*

Then (ii) *follows from* (i). *If the function $\ln^{-1/2}(e/t)\varphi(t)/t$ decreases, then the inverse assertion holds*: (i) *follows from* (ii).

Proof. First, assume that (i) holds. Since $L_{N_2} \subset M(\varphi)$, Corollary 10.2 implies that $\operatorname{Sym}(\mathcal{R}, M(\varphi)) = (M(\varphi))_{\ln^{1/2}}$. Moreover, $\operatorname{Sym}(\mathcal{R}, M(\varphi))$ is a symmetric space due to Proposition 10.2, such that its fundamental functions are defined as follows:

$$\varphi_{\operatorname{Sym}(\mathcal{R},M(\varphi))}(t) \asymp \varphi_{(M(\varphi))_{\ln^{1/2}}}(t) \asymp \Phi(t), \quad \text{where} \quad \Phi(t) = \sup_{0<s\le t} \frac{s \ln^{1/2}(e/s)}{\varphi(s)}. \tag{10.17}$$

It is clear that $\Phi(t)$ increases and $\Phi(t)/t$ decreases, i.e., this function is quasiconcave. Therefore, we can consider the Marcinkiewicz space $M(t/\Phi)$. From (10.17), we see that $(M(\varphi))_{\ln^{1/2}} \subset M(t/\Phi)$.

To prove the inverse embedding, we take $x \in M(t/\Phi)$, $\|x\| = 1$. Then $\int_0^t x^*(s)\, ds \le t/\Phi(t)$ for $0 < t \le 1$. Therefore, applying the inequality $t \ln^{1/2}(e/t)/\varphi(t) \le \Phi(t)$ and condition (i), we see that

$$\int_0^t x^*(s) \ln^{1/2}(e/s)\, ds = \int_0^t \ln^{1/2}(e/s)\, d\Big(\int_0^s x^*(u)\, du\Big) \le \int_0^t \ln^{1/2}(e/s)\, d\big(s/\Phi(s)\big)$$

$$= \ln^{1/2}(e/s)\, \frac{s}{\Phi(s)}\Big|_0^t + \frac{1}{2} \int_0^t \frac{\ln^{-1/2}(e/s)}{\Phi(s)}\, ds \le \varphi(t) + \frac{1}{2} \int_0^t \frac{\varphi(s)}{s} \ln^{-1}(e/s)\, ds \le (1 + C/2)\, \varphi(t).$$

This implies that $x \in (M(\varphi))_{\ln^{1/2}}$ and $\|x\|_{(M(\varphi))_{\ln^{1/2}}} \le 1 + C/2$.

Now, let (ii) hold and the function $\ln^{-1/2}(e/t)\varphi(t)/t$ decrease. Then, since $\Phi(t) = t \ln^{1/2}(e/t)/\varphi(t)$, it follows from (10.17) that

$$\operatorname{Sym}(\mathcal{R}, M(\varphi)) = M(\varphi(\cdot) \ln^{-1/2}(e/\cdot)). \tag{10.18}$$

Denote the least concave majorant of the function $\varphi(t)\log^{-1/2}(e/t)$ by $\psi(t)$. Then by [92, Theorem 2.1.1] the following inequality holds:

$$\varphi(t)\log^{-1/2}(e/t) \le \psi(t) \le 2\varphi(t)\log^{-1/2}(e/t), \quad 0 < t \le 1. \tag{10.19}$$

In particular, this implies that $\lim_{t\to 0}\psi(t) = 0$; therefore, the function ψ is absolutely continuous (see [92, Lemma 2.1.1]). Integrating by parts and taking into account (10.19), we obtain the inequality

$$\int_0^t \psi'(s)\ln^{1/2}(e/s)\,ds \ge \frac{1}{2}\int_0^t \frac{\psi(s)}{s}\ln^{-1/2}(e/s)\,ds \ge \frac{1}{2}\int_0^t \frac{\varphi(s)}{s}\ln^{-1}(e/s)\,ds. \tag{10.20}$$

Due to (10.18) and (10.19), we have $\operatorname{Sym}(\mathcal{R}, M(\varphi)) = M(\psi)$. On the other hand, $\operatorname{Sym}(\mathcal{R}, M(\varphi)) = (M(\varphi))_{\ln^{1/2}}$ by Corollary 10.2. Hence, $M(\psi) = (M(\varphi))_{\ln^{1/2}}$. Therefore, from $\psi' \in M(\psi)$, we see that $\psi' \in (M(\varphi))_{\ln^{1/2}}$. The last assertion means that

$$\int_0^t \psi'(s)\ln^{1/2}(e/s)\,ds \le C\,\varphi(t).$$

Thus, (i) follows from inequality (10.20). □

Remark 10.3. It follows from relation (i) of Theorem 10.4 that the function φ' belongs to the space $L\log\log L$. Indeed,

$$\int_0^1 \varphi'(s)\ln\ln(e/s)\,ds = \int_0^1 \varphi'(t)\int_t^1 \frac{ds}{s\ln(e/s)}\,dt$$
$$= \int_0^1 \frac{1}{s}\int_0^s \varphi'(t)\,dt\,\ln^{-1}(e/s)\,ds = \int_0^1 \frac{\varphi(s)}{s}\ln^{-1}(e/s)\,ds \le C\,\varphi(1).$$

Remark 10.4. The proof of Theorem 10.4 shows that

$$\operatorname{Sym}(\mathcal{R}, M(\varphi)) \subset M(t/\Phi(\cdot)),$$

where the function $\Phi(t)$ is defined as in (10.17).

Remark 10.5. Not any space $\operatorname{Sym}(\mathcal{R}, M(\varphi))$ is a Marcinkiewicz space. Indeed, if $\varphi(t) = 1$, then $M(\varphi) = L_1$ and $\operatorname{Sym}(\mathcal{R}, L_1)$ coincides with the Lorentz space $L\log^{1/2}L = \Lambda(t\ln^{1/2}(e/t))$. A similar result holds for every function φ such that $\ln^{-1/2}(e/t)\varphi(t)/t$ decreases and $\varphi' \notin L\log\log L$. The function $\varphi(t) = (\ln\ln(e^e/t))^{-1}$ can be taken as an example.

Corollary 10.3. *Let φ be a quasiconcave function with lower dilation index $\gamma_\varphi > 0$. Then $\operatorname{Sym}(\mathcal{R}, M(\varphi))$ is a Marcinkiewicz space.*

Proof. It suffices to apply Theorem 10.4 because

$$\int_0^t \frac{\varphi(s)}{s}\ln^{-1}(e/s)\,ds \le \int_0^t \frac{\varphi(s)}{s}\,ds \le C\,\varphi(t)$$

due to [92, Corollary 3, p. 80]. □

Remark 10.6. The condition of Corollary 10.3 is not necessary. Consider a function $\varphi(t) = \ln^{-1/p}(e/t)$ where $p > 0$. It is quasiconcave and direct calculations show that $\gamma_\varphi = 0$. One can easily check that

$$\int_0^t \frac{\ln^{-1/p}(e/s)}{s}\ln^{-1}(e/s)\,ds \asymp \ln^{-1/p}(e/t).$$

Therefore, $\operatorname{Sym}(\mathcal{R}, M(\varphi))$ is a Marcinkiewicz space by Theorem 10.4. Moreover, $\operatorname{Sym}(\mathcal{R}, M(\varphi)) = M(\psi)$ due to (10.17), where $\psi(t) = (\ln(e/t))^{-1/2-1/p}$.

Let us pass to the study of the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$. First of all, we are interested in the conditions of its symmetry. Theorem 10.1 shows that this is possible only if the lower dilation index of the fundamental function is equal to zero for the space X, i.e., X is quite "close" to L_∞. In particular, $\Lambda(\mathcal{R}, X)$ can coincide with L_∞, which is the least symmetric space on $[0,1]$. We find the characterization of such symmetric space X and show that the same conditions are necessary and sufficient for the relation $\operatorname{Sym}(\mathcal{R}, X) = L_\infty$. It is interesting to compare the result below with the Rodin–Semenov theorem (Theorem 3.2).

Theorem 10.5. *Let X be a symmetric space on $[0,1]$. Then the following conditions are equivalent to each other*:

(i) $\Lambda(\mathcal{R}, X) = L_\infty$;
(ii) $\operatorname{Sym}(\mathcal{R}, X) = L_\infty$;
(iii) $\ln^{1/2}(e/t) \notin X^\circ$.

We need two lemmas for the proof.

Lemma 10.3. *Let X be a symmetric space on $[0,1]$ such that $\ln^{1/2}(e/t) \notin X^\circ$. Then there exists a constant $c > 0$ depending only on X such that for every $\eta \in (0,1]$ there exists $\delta_0 \in (0,\eta)$ such that*

$$\frac{\| \ln^{1/2}(e/t)\chi_{[\delta,\eta]}\|_X}{\| \ln^{1/2}(e/t)\chi_{[\delta,1]}\|_X} \geq c \quad \textit{for all } \delta \in (0, \delta_0).$$

Proof. First, we show that

$$\gamma := \inf_{0<\eta\leq 1} \lim_{\delta\to 0^+} \| \ln^{1/2}(e/t)\chi_{[\delta,\eta]}\|_X > 0. \tag{10.21}$$

Indeed, if we assume that $\gamma = 0$, then we can construct a sequence $\{\eta_n\}$ decreasing to zero and such that

$$\| \ln^{1/2}(e/t)\chi_{[\eta_{n+1},\eta_n]}\|_X \leq 2^{-n}, \quad n = 1, 2, \ldots$$

Since X is a Banach space, it follows that $\ln^{1/2}(e/t) \in X$ and

$$\lim_{n\to\infty} \| \ln^{1/2}(e/t)\chi_{[0,\eta_n]}\|_X = 0.$$

Therefore, $\ln^{1/2}(e/t) \in X^\circ$, which contradicts the condition. Therefore, (10.21) is proved.

Assign $\alpha := \lim_{\delta\to 0^+} \| \ln^{1/2}(e/t)\chi_{[\delta,1]}\|_X$. If $\alpha < \infty$, then for a given $\eta \in (0,1]$, due to (10.21), we have the inequality $\| \ln^{1/2}(e/t)\chi_{[\delta,\eta]}\|_X \geq \gamma/2$ provided that $\delta > 0$ is sufficiently small. Therefore,

$$\frac{\| \ln^{1/2}(e/t)\chi_{[\delta,\eta]}\|_X}{\| \ln^{1/2}(e/t)\chi_{[\delta,1]}\|_X} \geq \frac{\gamma}{2\alpha}$$

for the specified δ.

If $\alpha = \infty$, then for $0 < \delta < \eta \leq 1$, we have

$$\begin{aligned}\frac{\| \ln^{1/2}(e/t)\chi_{[\delta,\eta]}\|_X}{\| \ln^{1/2}(e/t)\chi_{[\delta,1]}\|_X} &\geq \frac{\| \ln^{1/2}(e/t)\chi_{[\delta,1]}\|_X - \| \ln^{1/2}(e/t)\chi_{[\eta,1]}\|_X}{\| \ln^{1/2}(e/t)\chi_{[\delta,1]}\|_X} \\ &= 1 - \frac{\| \ln^{1/2}(e/t)\chi_{[\eta,1]}\|_X}{\| \ln^{1/2}(e/t)\chi_{[\delta,1]}\|_X} \geq \frac{1}{2},\end{aligned}$$

where $\eta \in (0,1]$ is fixed and $\delta > 0$ is sufficiently small. Therefore, the needed inequality is true in both cases and lemma is proved. $\square$

In the following lemma, we need the $\mathcal{K}$-functional

$$\kappa_a(t) = \mathcal{K}(t, a; \ell_1, \ell_2) \quad (t > 0)$$

generated by a Banach couple (ℓ_1, ℓ_2) of sequences. Recall the Holmstedt inequality from Proposition 2.2:

$$\frac{1}{4}\left\{\sum_{i=1}^{[t^2]} a_i^* + t\left[\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\right]^{1/2}\right\} \le \kappa_a(t) \le \left\{\sum_{i=1}^{[t^2]} a_i^* + t\left[\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\right]^{1/2}\right\}, \tag{10.22}$$

where $(a_i^*)_{i=1}^{\infty}$ is a nonincreasing permutation of the sequence $(|a_n|)_{n=1}^{\infty}$.

Lemma 10.4. *Let $b^n = 1/n \sum\limits_{k=1}^{n} e_k$ $(n \in \mathbb{N})$, where e_k are vectors of the canonical basis in a sequence space. Then there exist constants not depending on $t > 0$ and $n \in \mathbb{N}$ such that*

$$\kappa_{b^n}(t) \asymp \min\left(1, \frac{t}{\sqrt{n}}\right). \tag{10.23}$$

Proof. First, let $t \in \mathbb{N}$. Then, since $t^2 \ge n$, it follows from (10.22) that (10.23) holds. Now, let $t^2 < n$. Then, due to (10.22), we have

$$\alpha^{-1}(v^2 + v(1-v^2)^{1/2}) \le \kappa_{b^n}(t) \le v^2 + v(1-v^2)^{1/2},$$

where $v = t/\sqrt{n} < 1$. If $v \le 1/\sqrt{2}$, then this implies the inequality

$$\frac{v}{\alpha\sqrt{2}} \le \kappa_{b^n}(t) \le v^2 + v \le \left(1 + \frac{1}{\sqrt{2}}\right)v.$$

If $1/\sqrt{2} < v < 1$, then

$$\frac{v}{\alpha\sqrt{2}} \le \frac{v^2}{\alpha} \le \kappa_{b^n}(t) \le 2v.$$

Therefore, (10.23) is proved for all positive integers t. Therefore, due to the quasiconcavity of the $\mathcal{K}$-functional with respect to t, it is proved for all $t \ge 1$. It remains to note that for $0 < t \le 1$, by the definition of the $\mathcal{K}$-functional, we have $\kappa_{b^n}(t) = t\|b^n\|_2 = v$. $\square$

Proof of Theorem 10.5. The implication (i) $\Rightarrow$ (ii) follows from the embeddings

$$L_\infty \subset \mathrm{Sym}\,(\mathcal{R}, X) \subset \Lambda(\mathcal{R}, X)$$

which hold for any symmetric space X.

(ii) $\Rightarrow$ (iii). Assume that $\ln^{1/2}(e/t) \in X^\circ$. Then, due to (10.1), $L_{N_2} \subset X^\circ$; therefore, $\mathcal{R}(X) = \ell_2$ by Theorem 3.2. Therefore, by Corollary 10.1, we obtain that there exist constants not depending on $0 < a \le 1$ such that

$$\|\chi_{[0,a]}\|_{\mathrm{Sym}\,(\mathcal{R},X)} \asymp \sup_{\|(a_n)\|_2 \le 1} \left\|\chi_{[0,a]}\Big(\sum_{n=1}^{\infty} a_n r_n\Big)^*\right\|_X. \tag{10.24}$$

If $\|(a_n)\|_2 \le 1$, then it follows from (10.2) that

$$\Big(\sum_{n=1}^{\infty} a_n r_n\Big)^*(t) \le C \ln^{1/2}(e/t) \quad (0 < t \le 1),$$

whence

$$\chi_{[0,a]}(t)\Big(\sum_{n=1}^{\infty} a_n r_n\Big)^*(t) \le C\,\chi_{[0,a]}(t) \ln^{1/2}(e/t) \quad 0 < t, a \le 1.$$

Then, due to (10.24), the following inequality holds:

$$\|\chi_{[0,a]}\|_{\mathrm{Sym}\,(\mathcal{R},X)} \leq C\,\|\chi_{[0,a]} \ln^{1/2}(e/t)\|_X. \tag{10.25}$$

Since X° is separable, it follows that the norm of any function from this space is absolutely continuous. In particular, since $\ln^{1/2}(e/t) \in X^\circ$, it follows that

$$\|\chi_{[0,a]} \ln^{1/2}(e/t)\|_X \to 0 \quad \text{as} \quad a \to 0.$$

Therefore, from (10.25), it follows that $\lim\limits_{a\to 0} \|\chi_{[0,a]}\|_{\mathrm{Sym}\,(\mathcal{R},X)} = 0$. Thus, $\mathrm{Sym}\,(\mathcal{R}, X) \neq L_\infty$, and the implication is proved.

(iii) $\Rightarrow$ (i). First, prove the existence of a constant $c_1 > 0$ depending only on X such that for every integer $m \geq 0$ there exist $n_0 \geq 1$ such that for all $n \geq n_0$ and a dyadic interval Δ of rank m the following inequality holds:

$$\frac{\|\chi_\Delta \sum\limits_{m+1}^{m+n} r_i\|_X}{\|\sum\limits_{m+1}^{m+n} r_i\|_X} \geq c_1. \tag{10.26}$$

Note that, due to equimeasurability of the functions

$$\chi_\Delta(t) \sum_{i=m+1}^{m+n} r_i(t) \quad \text{and} \quad \chi_{[0,2^{-m}]}(t) \sum_{i=m+1}^{m+n} r_i(t),$$

it suffices to check relation (10.26) for $\Delta = [0, 2^{-m}]$.

For any $m \geq 0$ and $n \geq 1$, set

$$x_{m,n}(t) := \frac{1}{n} \sum_{m+1}^{m+n} r_i(t) \qquad \text{and} \qquad y_{m,n}(t) := x_{m,n}(t)\cdot\chi_{[0,2^{-m}]}(t).$$

Then, by the definition of Rademacher functions, we have

$$y^*_{m,n}(t) = x^*_{m,n}(2^m t)\cdot\chi_{[0,2^{-m}]}(t), \quad 0 < t \leq 1. \tag{10.27}$$

Moreover, due to Corollary 2.2, there exist universal constants $\beta \in (0, 1]$, $C_1 > 0$, and $C_2 > 0$ such that the following inequalities hold for a Rademacher sum $f_b := \sum\limits_1^\infty b_k r_k$:

$$f^*_b(t) \leq C_1 \kappa_b(\ln^{1/2}(e/t))$$

and

$$f^*_b(\beta t) \geq C_2^{-1} \kappa_b(\ln^{1/2}(e/t)).$$

Therefore, if $t \in (0, 1]$, then

$$x^*_{m,n}(t) \leq C_1 \kappa_{b^n_m}(\ln^{1/2}(e/t)) \tag{10.28}$$

and

$$x^*_{m,n}(\beta t) \geq C_2^{-1} \kappa_{b^n_m}(\ln^{1/2}(e/t)), \tag{10.29}$$

where $b^n_m = 1/n \sum\limits_{m+1}^{m+n} e_k$ ($\{e_k\}_{k=1}^\infty$ is the canonical basis in the space of sequences). Since $\kappa_{b^n_m}(t) = \kappa_{b^n}(t)$ ($t > 0$), it follows from Lemma 10.4 that

$$\kappa_{b^n_m}(t) \asymp \min\left(1, \frac{t}{\sqrt{n}}\right),$$

where constants do not depend on $t > 0$ and $n \geq 1$. Therefore, from (10.27)–(10.29), we obtain the inequality

$$x^*_{m,n}(t) \leq C_3 G_n(t), \tag{10.30}$$

where

$$G_n(t) := \min\Big\{1, \frac{\ln^{1/2}(e/t)}{\sqrt{n}}\Big\}, \quad 0 < t \le 1,$$

and

$$y^*_{m,n}(t) \ge C_4^{-1} F_{m,n}(t), \tag{10.31}$$

where

$$F_{m,n}(t) := \min\Big\{1, \frac{\ln^{1/2}(2^{1-m}\beta/t)}{\sqrt{n}}\Big\}, \quad 0 < t \le 2^{-m}\beta.$$

Now, we prove that there exists $c_2 > 0$ such that for every $m \ge 0$ the inequality

$$\|F_{m,n}\|_X \ge c_2 \|G_n\|_X \tag{10.32}$$

holds for any sufficiently large $n \in \mathbb{N}$. We show that if $m \ge 0$, then

$$F_{m,n}(t) \ge \frac{1}{2}\, G_n(t) \quad \text{ if } n \ge \frac{4m}{3} + \frac{4}{3}\log_2(1/\beta) \text{ and } 0 < t < 2^{-4m/3}\beta^{4/3}. \tag{10.33}$$

If $0 < t \le 2^{1-m} e^{-n}\beta$, then the last inequality is obvious because $F_{m,n}(t) = G_n(t) = 1$. If $2^{1-m}e^{-n}\beta < t \le e^{1-n}$, then inequality (10.33) takes the form

$$\ln^{1/2}(2^{1-m}\beta/t) \ge \frac{1}{2}\sqrt{n},$$

which holds for $n \ge \dfrac{4m}{3} + \dfrac{4}{3}\log_2(1/\beta)$. If $t > e^{1-n}$, then (10.33) is equivalent to the inequality

$$\ln^{1/2}(2^{1-m}\beta/t) \ge \frac{1}{2}\ln^{1/2}(e/t),$$

which holds for $0 < t < 2^{-4m/3}\beta^{4/3}$. Therefore, (10.33) is proved. Therefore, for $n \ge \frac{4m}{3} + \frac{4}{3}\log_2(1/\beta)$, we obtain the inequality

$$\|F_{m,n}\|_X \ge \frac{1}{2}\|G_n \chi_{[0,c_3 2^{-4m/3}]}\|_X, \tag{10.34}$$

where $c_3 = \beta^{4/3}$. This and the definition of G_n imply that the following inequality holds for $n \ge \frac{4m}{3} + \frac{4}{3}\log_2(1/\beta)$:

$$\|F_{m,n}\|_X \ge \frac{1}{2\sqrt{n}}\|\ln^{1/2}(e/t)\chi_{[e^{1-n},c_3 2^{-4m/3}]}\|_X. \tag{10.35}$$

On the other hand, due to Lemma 10.3, there exists a constant $c > 0$ such that for all $n \ge n_1(m)$, the following inequality holds:

$$\|\ln^{1/2}(e/t)\chi_{[e^{1-n},c_3 2^{-4m/3}]}\|_X \ge c\|\ln^{1/2}(e/t)\chi_{[e^{1-n},1]}\|_X.$$

From the last inequality and (10.35), it follows that

$$\|F_{m,n}\|_X \ge \frac{c}{2}\|G_n\chi_{[e^{1-n},1]}\|_X$$

for all $n \ge n_2(m) := \max\{\frac{4m}{3} + \frac{4}{3}\log_2(1/\beta), n_1(m)\}$ Therefore, taking into account (10.34), we conclude that (10.32) holds for $c_2 = c/2$ and $n \ge n_2(m)$.

Finally, the inequality

$$\frac{\|\chi_{[0,2^{-m}]}\sum\limits_{m+1}^{m+n} r_i\|_X}{\|\sum\limits_{m+1}^{m+n} r_i\|_X} = \frac{\|y_{m,n}\|_X}{\|x_{m,n}\|_X} \ge \frac{\|F_{m,n}\|_X}{C_3 C_4 \|G_n\|_X} \ge c_2 C_3^{-1} C_4^{-1}$$

holds for every $m \ge 0$ and for all $n \ge n_2(m)$ due to (10.30)–(10.32). Thus, (10.26) is proved with $c_1 = c_2 C_3^{-1} C_4^{-1}$.

Let us prove a similar relation for an arbitrary set $D \subset [0,1]$ of positive measure.

First of all, by the Lebesgue theorem on density points of measurable sets (see [110, Theorem 9.6.1]) for sufficiently large $m \in \mathbb{N}$, there exists a dyadic interval $\Delta := \Delta_m^{k_0} = [(k_0-1)2^{-m}, k_0 2^{-m}]$ such that

$$2^{-m} = m(\Delta) \geq m(\Delta \cap D) > 2^{-m-1}.$$

Consider the set $E = \cup_{k=1}^{2^m} E_m^k$ where E_m^k is obtained by translation of the set $\Delta \cap D$ on the interval Δ_m^k ($k = 1, 2, \ldots, 2^m$) (in particular, $E_m^{k_0} = \Delta \cap D$). Introduce $f_i = r_i \cdot \chi_E$, $i \in \mathbb{N}$. One can easily see that $|f_i(t)| \leq 1$, $t \in [0,1]$, $\|f_i\|_{L_2} \geq 1/\sqrt{2}$, and $f_i \to 0$ weakly in $L_2([0,1])$ as $i \to \infty$. Therefore, by Theorem 9.1, the sequence $\{f_i\}_{i=1}^{\infty}$ contains a subsequence $\{f_{i_j}\}$ equivalent in distribution to the Rademacher system. In other words, there exists $C > 0$ such that

$$\begin{aligned} C^{-1} m\Big\{t \in [0,1] : \Big|\sum_{j=1}^{l} a_j r_j(t)\Big| > Cz\Big\} &\leq m\Big\{t \in [0,1] : \Big|\sum_{j=1}^{l} a_j f_{i_j}(t)\Big| > z\Big\} \\ &\leq Cm\Big\{t \in [0,1] : \Big|\sum_{j=1}^{l} a_j r_j(t)\Big| > C^{-1}z\Big\} \end{aligned}$$

for all $l \in \mathbb{N}$, $a_j \in \mathbb{R}$ ($j = 1, 2, \ldots, l$), and $z > 0$. Hence, by the definition of r_j and f_j, for every $n \in \mathbb{N}$, the following inequality holds:

$$C^{-1} m\Big\{t \in [0,1] : \Big|\sum_{j=m+1}^{m+n} r_j(t)\chi_{[0,2^{-m}]}(t)\Big| > Cz\Big\}$$

$$\leq m\Big\{t \in [0,1] : \Big|\sum_{j=m+1}^{m+n} f_{i_j}(t)\chi_{\Delta}(t)\Big| > z\Big\} \leq Cm\Big\{t \in [0,1] : \Big|\sum_{j=m+1}^{m+n} r_j(t)\chi_{[0,2^{-m}]}(t)\Big| > C^{-1}z\Big\},$$

whence

$$\Big\|\sum_{j=m+1}^{m+n} r_{i_j}\chi_{\Delta\cap D}\Big\|_X \geq c_4 \Big\|\sum_{j=m+1}^{m+n} r_j \chi_{[0,2^{-m}]}\Big\|_X,$$

where $c_4 > 0$ depends only on C and X. Therefore, applying (10.26) to the dyadic interval $[0, 2^{-m}]$, we obtain that the following inequality holds for sufficiently large n :

$$\Big\|\sum_{j=m+1}^{m+n} r_{i_j}\chi_{\Delta\cap D}\Big\|_X \geq c_1 c_4 \Big\|\sum_{j=m+1}^{m+n} r_j\Big\|_X = c_1 c_4 \Big\|\sum_{j=m+1}^{m+n} r_{i_j}\Big\|_X.$$

This implies the inequality

$$\|\chi_D\|_{\Lambda(\mathcal{R},X)} \geq \|\chi_{D\cap D}\|_{\Lambda(\mathcal{R},X)} \geq c_1 c_4.$$

Since $\Lambda(\mathcal{R}, X)$ is a Banach ideal space, it follows from the last inequality that $\Lambda(\mathcal{R}, X) \subset L_\infty$. Due to the fact that the inverse embedding holds, the theorem is proved. $\square$

Corollary 10.4. *If X is a symmetric space on $[0,1]$, then the following conditions are equivalent to each other*:

(i) $\Lambda(\mathcal{R}, X) = L_\infty$;
(ii) $\mathrm{Sym}\,(\mathcal{R}, X) = L_\infty$;
(iii) $\ln^{1/2}(e/t) \notin X$.

Corollary 10.5. *If X and Y are symmetric spaces such that $X \subset Y$, then $\Lambda(\mathcal{R}, X) \subset \Lambda(\mathcal{R}, Y)$ and $\mathrm{Sym}\,(\mathcal{R}, X) \subset \mathrm{Sym}\,(\mathcal{R}, Y)$.*

Proof. If $\ln^{1/2}(e/t) \notin X^\circ$, then $\Lambda(\mathcal{R}, X) = L_\infty$ by Theorem 10.5; therefore, $\Lambda(\mathcal{R}, X) \subset \Lambda(\mathcal{R}, Y)$. Assume that $\ln^{1/2}(e/t) \in X^\circ$. Then $\ln^{1/2}(e/t) \in X \subset Y$. Therefore, $\mathcal{R}(X) = \mathcal{R}(Y) = \ell^2$ due to Lemma 3.2 and Theorem 3.2. Therefore,

$$\|x\|_{\Lambda(\mathcal{R},Y)} \asymp \sup\Big\{\Big\|x\sum_{n=1}^{\infty} a_n r_n\Big\|_Y : \|(a_n)\|_2 \le 1\Big\} \le C\cdot \sup\Big\{\Big\|x\sum_{n=1}^{\infty} a_n r_n\Big\|_X : \|(a_n)\|_2 \le 1\Big\} \asymp \|x\|_{\Lambda(\mathcal{R},X)},$$

and the first embedding is proved. The second one follows from the first embedding and the definition of the space $\mathrm{Sym}\,(\mathcal{R}, X)$. ☐

In the following corollary, we provide particular examples of symmetric spaces X such that $\Lambda(\mathcal{R}, X) = L_\infty$.

Corollary 10.6. *Let $\alpha \ge 2$ and X be either the Lorentz space $\Lambda(\varphi_\alpha)$, where $\varphi_\alpha(s) = \ln^{-1/\alpha}(e/s)$, or the exponential Orlicz space L_{N_α}, where $N_\alpha(t) = e^{t^\alpha} - 1$. Then $\Lambda(\mathcal{R}, X) = L_\infty$.*

Proof. Due to Theorem 10.5, it suffices to check that $\ln^{1/2}(e/t) \notin X^\circ$ in both cases. ☐

Theorem 10.5 can lead to the hypothesis that L_{N_2} is the largest symmetric space X such that $\Lambda(\mathcal{R}, X) = \mathrm{Sym}\,(\mathcal{R}, X) = L_\infty$. We provide a counterexample.

Example 10.4. Let ψ be a piecewise-linear function with knots at points $(t_k, \ln^{-1/2}(e/t_k))$, where $t_k \in [0, 1]$, $t_0 = 1 > t_1 > t_2 > \ldots$, $t_k \downarrow 0$. Then ψ is a quasiconcave function satisfying the following conditions:

(1) $0 \le \psi(t) \le \ln^{-1/2}(e/t)$;
(2) $\psi(t_k) = \ln^{-1/2}(e/t_k)$ $(k \ge 0)$;
(3) $\inf_{0<t\le 1} \psi(t) \ln^{1/2}(e/t) = 0$.

Note that we can choose points $\{t_k\}$ to satisfy the last condition. Consider the Marcinkiewicz space $M(t/\psi)$. From (1) and (3), we see that $L_{N_2} \overset{\neq}{\subset} M(t/\psi)$. Due to (2), we see that

$$\limsup_{t\to 0} \psi(t) \ln^{1/2}(e/t) \ne 0.$$

Since $tx^*(t) \le \int_0^t x^*(s)\,ds$, it follows that $\ln^{1/2}(e/t) \notin (M(t/\psi))^\circ$ (see [92, Lemma 2.5.4]). Therefore, by Theorem 10.5, we have

$$\Lambda(\mathcal{R}, M(t/\psi)) = \mathrm{Sym}\,(\mathcal{R}, M(t/\psi)) = L_\infty.$$

Remark 10.7. A simple analysis of the proof of Theorem 10.5 shows that for $\Lambda(\mathcal{R}, X) = L_\infty$, the norm in $\Lambda(\mathcal{R}, X)$ of the characteristic function of any dyadic interval from $[0, 1]$ is attained (up to equivalence) at the "tail" of a suitable Rademacher sum. There exists a constant $M_1 > 0$ depending only on X such that

$$\sup\Big\{\Big\|\chi_\Delta \cdot \sum_{i=n+1}^{\infty} a_i r_i\Big\|_X : \Big\|\sum_{i=n+1}^{\infty} a_i r_i\Big\|_X \le 1\Big\} \ge M_1 \tag{10.36}$$

for all $n \ge 1$, where Δ is an arbitrary dyadic interval of rank n. On the other hand, if X is interpolation with respect to the Banach couple (L_∞, L_{N_2}), then, as is shown in the proof of from [23, Theorem 1], a relation similar to (10.36) holds for partial Rademacher sums as well, i.e., there exists a constant $M_2 > 0$ depending only on X such that

$$\sup\Big\{\Big\|\chi_\Delta \cdot \sum_{i=1}^{n} a_i r_i\Big\|_X : \Big\|\sum_{i=1}^{n} a_i r_i\Big\|_X \le 1\Big\} \ge M_2 \tag{10.37}$$

for all $n \ge 1$ and any dyadic interval Δ of rank n.

Note that, unlike (10.36), the last relation holds not for all spaces X such that $\Lambda(\mathcal{R}, X) = L_\infty$. An example is proved below.

Example 10.5. First, recall some constructions from Chap. 4. For given symmetric space X on $[0,1]$, we consider the space of sequences $E = E_X$ with the norm

$$\|(a_k)\|_E := \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X.$$

Since E is interpolation with respect to the $\mathcal{K}$-monotone couple (ℓ_1, ℓ_2) (see Corollary 4.2), by the Brudnyi–Kruglyak theorem (see Sec. 0.5 or [50, Theorem 15.1]), there exists Banach ideal space F of two-sided sequences such that $(\min\{1, 2^k\})_{-\infty}^{\infty} \in F$ and

$$E = (\ell_1, \ell_2)_F^{\mathcal{K}} \quad \text{and} \quad \|a\|_E \asymp \|(\kappa_a(2^k))_{k=-\infty}^{\infty}\|_F. \tag{10.38}$$

Then, since $Y := (L_\infty, L_{N_2})_F^K$, it follows from Corollary 4.1 that

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_Y \asymp \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X, \quad \text{where} \quad (a_k)_{k=1}^{\infty} \in \ell_2. \tag{10.39}$$

Moreover, since $Y \in \mathcal{I}(L_\infty, L_{N_2})$, it follows that $Y \neq X$ provided that $X \notin \mathcal{I}(L_\infty, L_{N_2})$.

Let X be the Lorentz space $\Lambda(\varphi_0)$ generated by the function $\varphi_0(t) := \log_2^{-1/2}(2/t)$ $(0 < t \le 1)$. Find (up to equivalence) a fundamental function φ_Y of the corresponding space Y.

First of all, by definition, we have

$$\varphi_Y(t) = \|(\mathcal{K}(2^k, \chi_{[0,t]}; L_\infty, L_{N_2}))_{k=-\infty}^{\infty}\|_F.$$

Since, due to Lemma 4.4 and Remark 4.4, we have

$$K(t, x; L_\infty, L_{N_2}) \asymp t \sup_{0<s<2^{1-t^2}} x^*(s) \log_2^{-1/2}(2/s) \quad (t > 0),$$

it follows that, acting as in the proof of Theorem 4.3, one can easily show that

$$\mathcal{K}(2^k, \chi_{[0,t]}; L_\infty, L_{N_2}) \asymp \begin{cases} 2^k \log_2^{-1/2}(2/t) & \text{if } k \le \frac{1}{2}\log_2\log_2(2/t) \\ 1 & \text{if } k > \frac{1}{2}\log_2\log_2(2/t). \end{cases}$$

In order to find the Banach ideal space F from (10.38) corresponding to the space $X = \Lambda(\varphi_0)$, we apply Corollary 2.2 again and obtain the relation

$$\begin{aligned} \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\Lambda(\varphi_0)} &= \int_0^1 \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^*(t)\, d\varphi_0(t) \\ &\asymp \int_0^1 \kappa_a(\log_2^{1/2}(2/t))\, d(\log_2^{-1/2}(2/t)) \asymp \sum_{i=0}^{\infty} \kappa_a(2^i) \cdot 2^{-i}, \end{aligned}$$

where $a = (a_k)_{k=1}^{\infty}$. Therefore, due to the previous relations, for all $0 < t \le 1/2$, we have

$$\begin{aligned} \varphi_Y(t) &\asymp \sum_{i=0}^{\infty} \mathcal{K}(2^i, \chi_{[0,t]}; L_\infty, L_{N_2}) \cdot 2^{-i} \\ &\asymp \log_2^{-1/2}(2/t) \sum_{i \le \frac{1}{2}\log_2\log_2(2/t)} 1 + \sum_{i > \frac{1}{2}\log_2\log_2(2/t)} 2^{-i} \\ &\asymp \log_2^{-1/2}(2/t) \log_2\log_2(2/t). \end{aligned}$$

From Corollary 10.6, it follows that $\Lambda(\mathcal{R}, X) = L_\infty$. At the same time, relation (10.37) does not hold. Indeed, due to (10.39), for arbitrary $n \geq 2$ and any real $a_1, a_2, \ldots, a_n$, we have

$$\Big\| \chi_{[0,2^{-n}]} \cdot \sum_{i=1}^n a_i r_i \Big\|_{\Lambda(\varphi_0)} = \Big| \sum_{i=1}^n a_i \Big| \varphi_{\Lambda(\varphi_0)}(2^{-n})$$

$$= \frac{\varphi_0(2^{-n})}{\varphi_Y(2^{-n})} \Big\| \chi_{[0,2^{-n}]} \cdot \sum_{i=1}^n a_i r_i \Big\|_Y \leq \frac{C}{\log_2 n} \Big\| \sum_{i=1}^n a_i r_i \Big\|_Y = \frac{C}{\log_2 n} \Big\| \sum_{i=1}^n a_i r_i \Big\|_{\Lambda(\varphi_0)},$$

whence

$$\sup \Big\{ \Big\| \chi_{[0,2^{-n}]} \cdot \sum_{i=1}^n a_i r_i \Big\|_{\Lambda(\varphi_0)} : \Big\| \sum_{i=1}^n a_i r_i \Big\|_{\Lambda(\varphi_0)} \leq 1 \Big\} \leq \frac{C}{\log_2 n}.$$

To conclude the chapter, we show that there is a rather wide class of Orlicz spaces for which the Rademacher multiplicator space is a symmetric space different from L_∞.

Let F be an increasing function on $(0, \infty)$ such that $F(0) = 0$. Denote by $\operatorname{Exp} L^F$ the Orlicz space generated by the function $e^{F(t)} - 1$, which is assumed to be convex for sufficiently large $t > 0$. Recall that

$$\|x\|_{\operatorname{Exp} L^F} \asymp \sup_{0 < s \leq 1} \frac{x^*(s)}{F^{-1}(\ln(e/s))} \tag{10.40}$$

by Lemma 3.2. We show that there are conditions for F such that the space $\Lambda(\mathcal{R}, X)$, $X = \operatorname{Exp} L^F$, is symmetric though it is not equal to L_∞. To do that, we need the following statement.

Lemma 10.5. *For every $1 \leq p < \infty$ and for arbitrary $f \in L_p[0,1]$ there exists a sequence $(a_n) \in \ell_2$, $\|(a_n)\|_2 \leq 1$, such that*

$$\|f\|_p \leq 3p^{-1/2} \Big\| f \sum_{n=1}^\infty a_n r_n \Big\|_p.$$

Proof. Find a finite-valued function g such that its constancy sets are dyadic intervals and the following inequalities hold: $|g| \leq |f|$ and $\|f\|_p \leq 3/2 \|g\|_p$. Let N be the greatest rank of its constancy intervals. Take $m \in \mathbb{N}$ such that $m - 1 \leq p < m$. Set $a_n = 0$ for $n \leq N$, $a_n = m^{-1/2}$ for $N+1 \leq n \leq N+m$, and $a_n = 0$ for $n > N+m$. Then $\|(a_n)\|_2 = 1$. Since, by the definition of Rademacher functions, there exists a set of measure 2^{1-m} such that $|\sum_{N+1}^{N+m} r_n(t)| = m$, it follows that

$$\Big\| \sum_{n=1}^\infty a_n r_n \Big\|_p \geq m^{1/2} 2^{(1-m)/p} \geq 2^{-1} p^{1/2}.$$

One can easily see that g and $\sum_{n=1}^\infty a_n r_n = \sum_{n>N} a_n r_n$ are independent. Therefore, using [86, Theorem 2.2.4], we obtain the relation

$$\int_0^1 \Big| g(t) \sum_{n=1}^\infty a_n r_n(t) \Big|^p dt = \int_0^1 |g(t)|^p \, dt \int_0^1 \Big| \sum_{n=1}^\infty a_n r_n(t) \Big|^p dt,$$

whence

$$\|f\|_p \leq \frac{3}{2} \|g\|_p \leq 3p^{-1/2} \|g\|_p \Big\| \sum_{n=1}^\infty a_n r_n \Big\|_p = 3p^{-1/2} \Big\| g \sum_{n=1}^\infty a_n r_n \Big\|_p \leq 3p^{-1/2} \Big\| f \sum_{n=1}^\infty a_n r_n \Big\|_p.$$

□

Now we are ready to prove the main result of the chapter.

Theorem 10.6. *Let a function $F(t)$ be convex or $F(t) = t^\alpha$, $\alpha > 0$. Moreover, assume that the function $F^{-1}(t)t^{-1/2}$ increases. Then $\Lambda(\mathcal{R}, \operatorname{Exp} L^F) = \operatorname{Exp} L_\Psi$, where the function Ψ is defined by the relation $\Psi^{-1}(t) = F^{-1}(t)t^{-1/2}$.*

Proof. If $F(t) = t^\alpha$, it follows that Ψ^{-1} is a power function. Otherwise, if $F(t)$ is convex, it follows that Ψ^{-1} is quasiconcave because it increases by the condition, while $\Psi^{-1}(t)/t = (F^{-1}(t)/t) \cdot t^{-1/2}$ decreases because $F^{-1}(t)$ is quasiconcave. Therefore, in the second case, Ψ is equivalent to a convex-increasing function. Thus, the norm of the space $\operatorname{Exp} L^\Psi$ has the following extrapolation description for both cases:

$$\|f\|_{\operatorname{Exp} L^\Psi} \asymp \sup_{1 \le p < \infty} \frac{\|f\|_p}{\Psi^{-1}(p)}. \tag{10.41}$$

Indeed, in the power case, it suffices to expand the function $e^{u^\beta} - 1$ into a Taylor series, where $\beta = 2\alpha/(2-\alpha) > 0$ by the condition. If $F(t)$ is convex, then we can apply [24, Corollary 1] (see also [21, Proposition 1]).

Now, we show that $\Lambda(\mathcal{R}, \operatorname{Exp} L^F) \subset \operatorname{Exp} L^\Psi$. By Lemma 10.5, for given $1 \le p < \infty$ and $f \in L_p$, there exists a Rademacher sum $h_p = \sum a_k r_k$ such that $\|(a_k)\|_2 \le 1$ and $\|f\|_p \le 3\, p^{-1/2}\, \|f h_p\|_p$. It follows from the assumption of the theorem that $F(t) \le C\, t^2$ $(t \ge 1)$. Therefore, $L_{N_2} \subset \operatorname{Exp} L^F$ and $\mathcal{R}(\operatorname{Exp} L^F) = \ell_2$ by Theorem 3.2. Therefore, using (10.41) and the relation $F^{-1}(t) = \Psi^{-1}(t)t^{1/2}$, we obtain the relation

$$\begin{aligned}
\|f\|_{\operatorname{Exp} L^\Psi} &\asymp \sup_{1 \le p < \infty} \frac{1}{\Psi^{-1}(p)}\, \|f\|_p \le 3 \sup_{1 \le p < \infty} \frac{p^{-1/2}}{\Psi^{-1}(p)}\, \|f h_p\|_p \\
&\le 3 \sup_{1 \le p < \infty} \frac{p^{-1/2}}{\Psi^{-1}(p)} \sup_{\|(a_k)\|_2 \le 1} \|f \sum a_k r_k\|_p = 3 \sup_{\|(a_k)\|_2 \le 1} \sup_{1 \le p < \infty} \frac{1}{F^{-1}(p)}\, \|f \sum a_k r_k\|_p \\
&\asymp \sup_{\|(a_k)\|_2 \le 1} \|f \sum a_k r_k\|_{\operatorname{Exp} L^F} \asymp \|f\|_{\Lambda(\mathcal{R}, \operatorname{Exp} L^F)}.
\end{aligned}$$

By Lemma 3.2 (see relation (10.40)), the space $\operatorname{Exp} L^F$ is the Marcinkiewicz space $M(\varphi)$ for $\varphi(t) = tF^{-1}(\ln^{1/2}(e/t))$. Moreover, calculations show that $\gamma_\varphi > 0$. Therefore, $\operatorname{Sym}(\operatorname{Exp} L^F)$ is a Marcinkiewicz space due to Corollary 10.3. Since $t^{1/2}/F^{-1}(t)$ decreases, it follows that, applying (10.40), we find a fundamental function of the last space:

$$\begin{aligned}
\varphi_{\operatorname{Sym}(\operatorname{Exp} L^F)}(t) &= \|\chi_{[0,t]} \ln^{1/2}(e/s)\|_{\operatorname{Exp} L^F} \\
&\asymp \sup_{0 < s \le t} \frac{\ln^{1/2}(e/s)}{F^{-1}(\ln(e/s))} = \frac{\ln^{1/2}(e/t)}{F^{-1}(\ln(e/t))} = \frac{1}{\Psi^{-1}(\ln(e/t))},
\end{aligned}$$

i.e., it is equivalent to the fundamental function of the space $\operatorname{Exp} L^\Psi$. By Lemma 3.2, it is a Marcinkiewicz space. Thus, we obtain that $\operatorname{Sym}(\operatorname{Exp} L^F) = \operatorname{Exp} L^\Psi$. Finally $\Lambda(\mathcal{R}, \operatorname{Exp} L^F) = \operatorname{Exp} L^\Psi$. □

Example 10.6. Consider the space $\operatorname{Exp} L_p = L_{N_{1/p}}$ (this is the Marcinkiewicz space generated by the function $\varphi_p(t) = t \ln^{1/p}(e/t)$). If $0 < p < 2$, then $\Lambda(\mathcal{R}, \operatorname{Exp} L_p) = \operatorname{Exp} L_q$ for $q = 2p/(2-p)$ by Theorem 10.6.

Comments and references. The notion of the Rademacher multiplicator space is connected in a sense with the theory of integration with respect to vector measures. The space $\Lambda(\mathcal{R}, L_1)$ appears as the space of functions integrable in the sense of Bartle, Dunford, and Schwartz with respect to the vector measure

$$\nu(A) = \Big(\int_A r_n(t)\, dt \Big)_{n=1}^{\infty} \quad (A \subset [0,1] \text{ is measurable in the Lebesgue sense})$$

valued in ℓ_2 (see [54]). Also, the space can be characterized as the set of all functions f such that for every measurable set $A \subset [0,1]$ the sequence of Fourier–Rademacher coefficients of $f\chi_A$ belongs to ℓ_2 (a description of Fourier–Rademacher coefficients for functions from common symmetric spaces is given in [19]). The definition of Rademacher multiplicator space (as well as initial results) appeared in [55]. Lemma 10.1 and Theorem 10.1 are proved in [27] (weaker variants of these statements were obtained in [55]). The symmetric kernel of the space of multiplicators is introduced ard studied in detail in [27]. One of the main results of this chapter is Theorem 10.5. It characterizes symmetric spaces X such that $\Lambda(\mathcal{R}, X) = L_\infty$ and $\mathrm{Sym}\,(\mathcal{R}, X) = L_\infty$. In [27], the equivalence (ii) $\Leftrightarrow$ (iii) is proved and in [28], the equivalence (i) $\Leftrightarrow$ (iii) is proved. Earlier, more restrictive sufficient conditions were found. In [56, 57], such condition is the boundedness in X of the operator

$$\mathcal{L}_2 f(t) = \left(\frac{1}{\varphi_1(t)} \int\limits_0^1 (f^*(s))^2 \, d\varphi_1(s) \right)^{1/2}, \quad \text{where} \quad \varphi_1(t) = \log_2^{-1}(2/t).$$

In [23] a stronger result was obtained: $\Lambda(\mathcal{R}, X) = L_\infty$ if the symmetric space X is interpolation with respect to the Banach couple (L_∞, L_{N_2}). Lemma 10.4 is taken from the last work. Theorem 10.6 is a rather unexpected application of the extrapolation description of exponential Orlicz spaces (10.41). Lemma 10.5 is proved in [55].

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